

# IRREDUCIBLE MODULES OVER KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF TYPE $A_n$ AND SEMISTANDARD TABLEAUX

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**ABSTRACT.** Using combinatorics of Young tableaux, we give an explicit construction of irreducible graded modules over Khovanov-Lauda-Rouquier algebras  $R$  and their cyclotomic quotients  $R^\lambda$  of type  $A_n$ . Our construction is compatible with crystal structure. Let  $\mathbf{B}(\infty)$  and  $\mathbf{B}(\lambda)$  be the  $U_q(\mathfrak{sl}_{n+1})$ -crystal consisting of marginally large tableaux and semistandard tableaux of shape  $\lambda$ , respectively. On the other hand, let  $\mathfrak{B}(\infty)$  and  $\mathfrak{B}(\lambda)$  be the  $U_q(\mathfrak{sl}_{n+1})$ -crystals consisting of isomorphism classes of irreducible graded  $R$ -modules and  $R^\lambda$ -modules, respectively. We show that there exist explicit crystal isomorphisms  $\Phi_\infty : \mathbf{B}(\infty) \xrightarrow{\sim} \mathfrak{B}(\infty)$  and  $\Phi_\lambda : \mathbf{B}(\lambda) \xrightarrow{\sim} \mathfrak{B}(\lambda)$ .

## INTRODUCTION

Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra and let  $U_q^-(\mathfrak{g})$  be the negative part of the quantum group  $U_q(\mathfrak{g})$  associated with  $\mathfrak{g}$ . Recently, Khovanov and Lauda [15, 16] and Rouquier [21] independently introduced a new family of graded algebras  $R$  whose representation theory gives a categorification of  $U_q^-(\mathfrak{g})$ . The algebra  $R$  is called the *Khovanov-Lauda-Rouquier algebra* associated with  $\mathfrak{g}$ . Let  $\lambda \in P^+$  be a dominant integral weight. It was conjectured that the *cyclotomic quotient*  $R^\lambda$  gives a categorification of irreducible highest weight  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  with highest weight  $\lambda$  [16]. This conjecture was shown to be true when  $\mathfrak{g}$  is of type  $A_\infty$  or  $A_n^{(1)}$  [1, 2, 3].

In [19], Lauda and Vazirani investigated the crystal structure on the set of isomorphism classes of finite dimensional irreducible graded modules over  $R$  and  $R^\lambda$ , where the Kashiwara operators are defined in terms of induction and restriction functors. Let  $\mathfrak{B}(\infty)$  and  $\mathfrak{B}(\lambda)$  denote the  $U_q(\mathfrak{g})$ -crystal consisting of irreducible graded  $R$ -modules and  $R^\lambda$ -modules, respectively. They showed that there exist  $U_q(\mathfrak{g})$ -crystal isomorphisms  $\mathfrak{B}(\infty) \xrightarrow{\sim} B(\infty)$  and  $\mathfrak{B}(\lambda) \xrightarrow{\sim} B(\lambda)$ , where  $B(\infty)$  and  $B(\lambda)$  are the crystals of  $U_q^-(\mathfrak{g})$  and  $V(\lambda)$ , respectively. Consequently, every irreducible graded module can be constructed *inductively* by applying the Kashiwara operators on the trivial module.

On the other hand, in [18], Kleshchev and Ram gave an explicit construction of irreducible graded  $R$ -modules for all finite type using combinatorics of Lyndon words. They characterized the irreducible graded  $R$ -modules as the simple heads of certain induced modules. In [7], Hill, Melvin and Mondragon

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constructed cuspidal representations for all finite type and completed the classification of irreducible graded  $R$ -modules given in [18]. It is still an open problem to construct irreducible graded  $R^\lambda$ -modules in terms of Lyndon words. However, in this approach, the action of Kashiwara operators is hidden in the combinatorics of Lyndon words.

In this paper, using combinatorics of Young tableaux, we give an explicit construction of irreducible graded  $R$ -modules and  $R^\lambda$ -modules when  $\mathfrak{g}$  is of type  $A_n$ . Our construction is compatible with crystal structure in the following sense. Let  $\mathbf{B}(\lambda)$  be the set of all *semistandard tableaux* of shape  $\lambda$  with entries in  $\{1, 2, \dots, n+1\}$  and let  $\mathbf{B}(\infty)$  be the set of all *marginally large tableaux*. It is well-known that  $\mathbf{B}(\lambda)$  and  $\mathbf{B}(\infty)$  have  $U_q(\mathfrak{sl}_{n+1})$ -crystal structures and they are isomorphic to  $B(\lambda)$  and  $B(\infty)$ , respectively [8, 9, 14, 20]. For each semistandard tableau of shape  $\lambda$  (resp. a marginally large tableau), we construct an irreducible graded  $R^\lambda$ -module (resp.  $R$ -module) and show that there exist explicit crystal isomorphisms  $\Phi_\lambda : \mathbf{B}(\lambda) \xrightarrow{\sim} \mathfrak{B}(\lambda)$  and  $\Phi_\infty : \mathbf{B}(\infty) \xrightarrow{\sim} \mathfrak{B}(\infty)$ . In our construction, irreducible graded modules appear as the simple heads of certain induced modules that are determined by semistandard tableaux or marginally large tableaux. Our work was inspired by [18] and [22]. We expect our work can be extended to other classical type using combinatorics of Kashiwara-Nakashima tableaux in [14]. As was shown in [10], one may construct irreducible modules over the Khovanov-Lauda-Rouquier algebra of type  $A$  using cellular basis technique introduced in [6].

This paper is organized as follows. In Section 1 and Section 2, we review the theory of  $U_q(\mathfrak{sl}_{n+1})$ -crystals and their combinatorial realization in terms of Young tableaux. In Section 3, we recall the fundamental properties of Khovanov-Lauda-Rouquier algebras  $R$  and their cyclotomic quotients  $R^\lambda$ . We also describe the crystal structures on  $\mathfrak{B}(\infty)$  and  $\mathfrak{B}(\lambda)$ . Section 4 is devoted to the main result of our paper. For each semistandard tableau  $T \in \mathbf{B}(\lambda)$ , we construct an irreducible graded  $R^\lambda$ -module  $\Phi_\lambda(T) := \text{hdInd}\nabla_T$  as the simple heads of the induced module  $\text{Ind}\nabla_T$  determined by  $T$ , and show that the correspondence  $T \mapsto \text{hdInd}\nabla_T$  defines a crystal isomorphism  $\Phi_\lambda : \mathbf{B}(\lambda) \xrightarrow{\sim} \mathfrak{B}(\lambda)$ . In Section 5, we extend the construction given in Section 4 to marginally large tableaux to obtain an explicit construction of irreducible graded  $R$ -modules. We also show that there exists an explicit crystal isomorphism  $\Phi_\infty : \mathbf{B}(\infty) \xrightarrow{\sim} \mathfrak{B}(\infty)$  induced by  $\Phi_\lambda$ .

## 1. THE CRYSTAL $B(\lambda)$ AND SEMISTANDARD TABLEAUX

In this section, we review the theory of  $U_q(\mathfrak{sl}_{n+1})$ -crystals and their connection with combinatorics of Young tableaux (see, for example, [8, 14]). Let  $I = \{1, 2, \dots, n\}$  and let

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be the Cartan matrix of type  $A_n$ . Set  $P^\vee = \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$ ,  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ , and define the linear functionals  $\alpha_i, \varpi_i \in \mathfrak{h}^*$  ( $i \in I$ ) by

$$\alpha_i(h_j) = a_{ji} \quad \varpi_i(h_j) = \delta_{ij} \quad (i, j \in I).$$

The  $\alpha_i$  (resp.  $\varpi_i$ ) are called the *simple roots* (resp. *fundamental weights*). Set  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ ,  $Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$  and  $P = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_n$ . The quadruple  $(A, P^\vee, \Pi, P)$  is called the *Cartan datum of type  $A_n$* . The free abelian groups  $P^\vee$ ,  $P$  and  $Q$  are called the *dual weight lattice*, *weight lattice*, and *root lattice*, respectively. We denote by  $P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I\}$  the set of all *dominant integral weights*. Define

$$\epsilon_1 = \varpi_1, \quad \epsilon_{k+1} = \varpi_{k+1} - \varpi_k \quad (k \geq 1).$$

Then  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ,  $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$ , and every dominant integral weight  $\lambda = a_1\varpi_1 + \cdots + a_n\varpi_n$  can be written as  $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$ , where  $\lambda_i = a_i + \cdots + a_n$  ( $i = 1, \dots, n$ ).

Let  $q$  be an indeterminate and for  $m \geq n \geq 0$ , define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q, \quad \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

**Definition 1.1.** The *quantum special linear algebra*  $U_q(\mathfrak{sl}_{n+1})$  is the associative algebra over  $\mathbb{C}(q)$  generated by the elements  $e_i, f_i$  ( $i = 1, \dots, n$ ) and  $q^h$  ( $h \in P^\vee$ ) with the following defining relations:

$$\begin{aligned} (1.1) \quad & q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee, \\ & q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \\ & e_i f_j - f_j e_i = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \\ & \sum_{r=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-r)} e_j e_i^{(r)} = \sum_{r=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-r)} f_j f_i^{(r)} = 0 \quad (i \neq j). \end{aligned}$$

Here, we use the notation  $e_i^{(k)} = e_i^k / [k]_q!$ ,  $f_i^{(k)} = f_i^k / [k]_q!$ .

For each  $\lambda \in P^+$ , there exists a unique irreducible highest weight  $U_q(\mathfrak{sl}_{n+1})$ -module  $V(\lambda)$  with highest weight  $\lambda$ . It was shown in [11, 12] that every irreducible highest weight module  $V(\lambda)$  has a crystal basis  $(L(\lambda), B(\lambda))$ . The crystal  $B(\lambda)$  can be thought of as a basis at  $q = 0$  and most of combinatorial features of  $V(\lambda)$  are reflected on the structure of  $B(\lambda)$ . Moreover, the crystal bases have very nice behavior with respect to tensor product. The basic properties of crystal bases can be found in [8, 11, 12], etc.

By extracting the standard properties of crystal bases, Kashiwara introduced the notion of abstract crystals in [13]. An *abstract crystal* is a set  $B$  together with the maps  $\text{wt} : B \rightarrow P$ ,  $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{0\}$ ,  $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$  ( $i \in I$ ) satisfying certain conditions. The details on abstract crystals, including the notion of strict morphism, embedding, isomorphism, etc., can be found in [8, 13]. We only give some examples including the tensor product of abstract crystals.

**Example 1.2.**

- (1) Let  $(L(\lambda), B(\lambda))$  be the crystal basis of the highest weight module  $V(\lambda)$  with highest weight  $\lambda \in P^+$ . Then  $B(\lambda)$  is a  $U_q(\mathfrak{sl}_{n+1})$ -crystal.
- (2) Let  $(L(\infty), B(\infty))$  be the crystal basis of  $U_q^-(\mathfrak{sl}_{n+1})$ . Then  $B(\infty)$  is a  $U_q(\mathfrak{sl}_{n+1})$ -crystal.
- (3) For  $\lambda \in P$ , let  $\mathbf{T}^\lambda = \{t_\lambda\}$  and define the maps

$$\begin{aligned} \text{wt}(t_\lambda) &= \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \text{ for } i \in I, \\ \varepsilon_i(t_\lambda) &= \varphi_i(t_\lambda) = -\infty \text{ for } i \in I. \end{aligned}$$

Then  $\mathbf{T}^\lambda$  is a  $U_q(\mathfrak{sl}_{n+1})$ -crystal.

- (4) Let  $\mathbf{C} = \{c\}$  and define the maps

$$\text{wt}(c) = 0, \quad \tilde{e}_i c = \tilde{f}_i c = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0 \quad (i \in I).$$

Then  $\mathbf{C}$  is a  $U_q(\mathfrak{sl}_{n+1})$ -crystal.

- (5) Let  $B_1, B_2$  be crystals and set  $B_1 \otimes B_2 = B_1 \times B_2$ . Define the maps

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle\}, \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

Then  $B_1 \otimes B_2$  is a  $U_q(\mathfrak{sl}_{n+1})$ -crystal.

We now recall the connection between the theory of  $U_q(\mathfrak{sl}_{n+1})$ -crystals and combinatorics of Young tableaux. A *Young diagram*  $\lambda$  is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. We denote by  $\mathcal{Y}$  the set of all Young diagrams. If a Young diagram  $\lambda$  contains  $N$  boxes, we write  $\lambda \vdash N$  and  $|\lambda| = N$ . The number of rows in  $\lambda$  will be denoted by  $l(\lambda)$ . We denote by  ${}^t\lambda$  denotes the Young diagram obtained by flipping  $\lambda$  over its main diagonal. We usually identify a Young diagram  $\lambda$  with the partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ , where  $\lambda_i$  is the number of boxes in the  $i$ th row of  $\lambda$ . Recall that a dominant integral weight  $\lambda = a_1\varpi_1 + \dots + a_n\varpi_n$  can be written as  $\lambda = \lambda_1\epsilon_1 + \dots + \lambda_n\epsilon_n$ , where  $\lambda_i = a_i + \dots + a_n$  ( $i = 1, \dots, n$ ). Since  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , we identify a dominant integral weight  $\lambda = a_1\varpi_1 + \dots + a_n\varpi_n$  with a partition  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ .

A *tableau*  $T$  of shape  $\lambda$  is a filling of a Young diagram  $\lambda$  with numbers, one for each box. We say that a tableau  $T$  is *semistandard* if

- (1) the entries in each row are weakly increasing from left to right,
- (2) the entries in each column are strictly increasing from top to bottom.

We denote by  $\mathbf{B}(\lambda)$  the set of all semistandard tableaux of shape  $\lambda$  with entries in  $\{1, 2, \dots, n+1\}$ .

Let  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0)$  be a Young diagram with  $l(\lambda) = s$  and  $|\lambda| = \lambda_1 + \dots + \lambda_s = N$ . It is well-known that  $\mathbf{B}(\lambda)$  has a  $U_q(\mathfrak{sl}_{n+1})$ -crystal structure and is isomorphic to the crystal  $B(\lambda)$ . Let us briefly recall how to define the crystal structure on  $\mathbf{B}(\lambda)$ . Let  $\mathbf{B} = \mathbf{B}(\varpi_1)$  be the crystal of the vector representation  $V(\varpi_1)$  given below.

$$\mathbf{B} : \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \dots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}.$$

By the *Middle-Eastern reading*, we mean the reading of entries of a semistandard tableau by moving across the rows from right to left and from top to bottom. Thus we get an embedding  $\Upsilon_M : \mathbf{B}(\lambda) \rightarrow \mathbf{B}^{\otimes N}$  and one can define a  $U_q(\mathfrak{sl}_{n+1})$ -crystal structure on  $\mathbf{B}(\lambda)$  by the inverse of  $\Upsilon_M$ . On the other hand, the *Far-Eastern reading* proceeds down the columns from top to bottom and from right to left and yields an embedding  $\Upsilon_F : \mathbf{B}(\lambda) \rightarrow \mathbf{B}^{\otimes N}$ , which also defines a  $U_q(\mathfrak{sl}_{n+1})$ -crystal structure on  $\mathbf{B}(\lambda)$ . It is known that the crystal structure on  $\mathbf{B}(\lambda)$  does not depend on  $\Upsilon_M$  or  $\Upsilon_F$  and that it is isomorphic to  $B(\lambda)$  (see, for example, [8]), where the highest weight vector is given by

$$T_\lambda = \begin{array}{|c|c|c|c|c|c|} \hline 1 & \dots & \dots & 1 & 1 & 1 \\ \hline 2 & \dots & \dots & 2 & & \\ \hline \vdots & \vdots & \vdots & & & \\ \hline s & \dots & s & & & \\ \hline \end{array}.$$

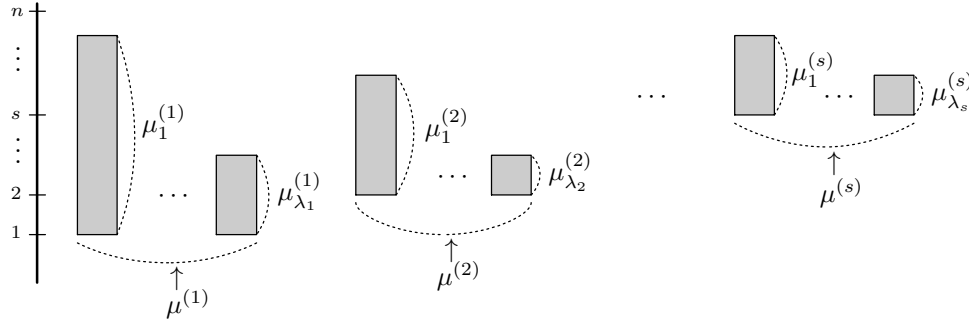
For a semistandard tableau  $T \in \mathbf{B}(\lambda)$ , write

$$\Upsilon_M(T) = \boxed{a_{1,\lambda_1}^T} \otimes \dots \otimes \boxed{a_{1,1}^T} \otimes \boxed{a_{2,\lambda_2}^T} \otimes \dots \otimes \boxed{a_{2,1}^T} \otimes \dots \otimes \boxed{a_{s,1}^T},$$

where  $a_{ij}^T$  is the entry in the  $j$ th box of the  $i$ th row of  $T$ . Define a map  $\Psi_\lambda : \mathbf{B}(\lambda) \rightarrow \mathcal{Y}^s$  by

$$(1.2) \quad \Psi_\lambda(T) := (\mu^{(1)}, \dots, \mu^{(s)}),$$

where  $\mu^{(k)} = (a_{k,\lambda_k}^T - k, a_{k,\lambda_k-1}^T - k, \dots, a_{k,1}^T - k)$  for  $k = 1, \dots, s$ . Note that  $\mu^{(k)}$  could be the empty Young diagram  $(0, 0, \dots)$  and that  $\Psi_\lambda$  is injective. Pictorially,  $\Psi_\lambda(T) = (\mu^{(1)}, \dots, \mu^{(s)})$  can be visualized as follows:



Here,  $\mu^{(i)} = (\mu_1^{(i)} \geq \mu_2^{(i)} \geq \dots \geq \mu_{\lambda_i}^{(i)} \geq 0)$  for  $i = 1, \dots, s$ .

**Example 1.3.** Let  $\mathfrak{g} = \mathfrak{sl}_6$  and  $\lambda = 2\varpi_1 + 2\varpi_2 + \varpi_4 + \varpi_5$ . If

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 3 & 3 & 4 & 6 \\ \hline 2 & 3 & 4 & 5 & & \\ \hline 3 & 5 & & & & \\ \hline 5 & 6 & & & & \\ \hline 6 & & & & & \\ \hline \end{array},$$

then

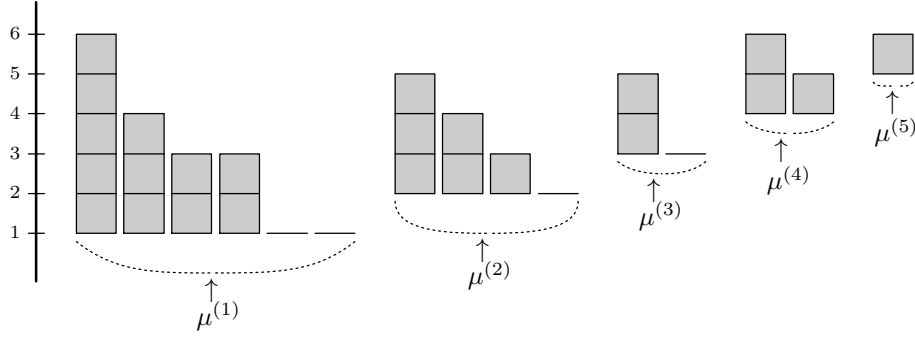
$$\Upsilon_M(T) = \boxed{6} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{1} \otimes \boxed{5} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{5} \otimes \boxed{3} \otimes \boxed{6} \otimes \boxed{5} \otimes \boxed{6},$$

$$\Upsilon_F(T) = \boxed{6} \otimes \boxed{4} \otimes \boxed{3} \otimes \boxed{5} \otimes \boxed{3} \otimes \boxed{4} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{5} \otimes \boxed{6} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{3} \otimes \boxed{5} \otimes \boxed{6},$$

and  $\Psi_\lambda(T) = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \mu^{(4)}, \mu^{(5)})$ , where

$$\mu^{(1)} := (5, 3, 2, 2, 0, 0), \quad \mu^{(2)} := (3, 2, 1, 0), \quad \mu^{(3)} := (2, 0), \quad \mu^{(4)} := (2, 1), \quad \mu^{(5)} := (1).$$

Pictorially,  $\Psi_\lambda(T) = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \mu^{(4)}, \mu^{(5)})$  is given as follows:



Note that  $\Upsilon_M(T)$  can be obtained by reading the top entries of columns in the above diagram from left to right.

The following lemma will play a crucial role in proving our main result (Theorem 4.8).

**Lemma 1.4.** *Let  $T$  be a semistandard tableau of shape  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0)$ , and let*

$$\Psi_\lambda(T) = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(s)}),$$

where  $\mu^{(i)} = (\mu_1^{(i)} \geq \mu_2^{(i)} \geq \dots \geq \mu_{\lambda_i}^{(i)} \geq 0)$  for  $i = 1, \dots, s$ . Suppose that  $T$  is not the highest weight vector  $T_\lambda$ ; i.e., not all  $\mu^{(1)}, \dots, \mu^{(s)}$  are  $(0, 0, \dots)$ . Set

$$i_T = \min\{\mu_j^{(i)} + i - 1 \mid 1 \leq i \leq s, 1 \leq j \leq \lambda_i, \mu_j^{(i)} > 0\},$$

$$\varepsilon = \varepsilon_{i_T}(T).$$

Then we have

$$(1) \quad \varepsilon_{i_T}(T) = \#\{\mu_j^{(i)} \mid \mu_j^{(i)} > 0, \mu_j^{(i)} + i - 1 = i_T, 1 \leq i \leq s, 1 \leq j \leq \lambda_i\};$$

(2)

$$\tilde{e}_{i_T}^\varepsilon(T) = T^+,$$

where  $T^+$  is the tableau of shape  $\lambda$  obtained from  $T$  by replacing all entries  $i_T + 1$  by  $i_T$  from the top row to the  $i_T$ th row.

*Proof.* Let

$$\Upsilon_M(T) = \boxed{a_{1,\lambda_1}} \otimes \cdots \otimes \boxed{a_{1,1}} \otimes \boxed{a_{2,\lambda_2}} \otimes \cdots \otimes \boxed{a_{2,1}} \otimes \cdots \otimes \boxed{a_{s,1}},$$

where  $a_{ij}$  is the entry in the  $j$ th box of the  $i$ th row of  $T$ . Then, from the definition of  $\Psi_\lambda$ , we have

$$a_{i,\lambda_i-j+1} = \mu_j^{(i)} + i \quad (1 \leq j \leq \lambda_i),$$

which yields

$$\begin{aligned} i_T &= \min\{a_{ij} - 1 \mid 1 \leq i \leq s, 1 \leq j \leq \lambda_i, a_{ij} > i\}, \\ \#\{a_{ij} \mid a_{ij} > i, a_{ij} - 1 = i_T\} &= \#\{\mu_j^{(i)} \mid \mu_j^{(i)} > 0, \mu_j^{(i)} + i - 1 = i_T\}. \end{aligned}$$

Note that the set  $\{a_{ij} - 1 \mid 1 \leq i \leq s, 1 \leq j \leq \lambda_i, a_{ij} > i\}$  is not empty since  $T$  is not the highest weight vector  $T_\lambda$ . Take the rightmost number  $a_{pq}$  of  $\Upsilon_M(T)$  such that  $a_{pq} = i_T + 1$ . Since  $T$  is semistandard; i.e.,

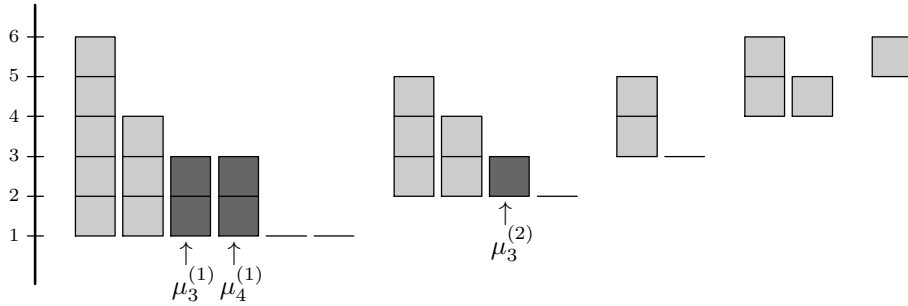
$$\begin{aligned} a_{pk} &\geq a_{pq} \quad \text{for } k \geq q, \\ a_{p'q'} &\neq a_{pq} - 1 \quad \text{for } 1 \leq p' < p, \end{aligned}$$

and

$$\varepsilon_i(\boxed{j}) = \begin{cases} 1 & \text{if } i = j - 1, \\ 0 & \text{otherwise,} \end{cases} \quad \varphi_i(\boxed{j}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

our assertion follows from the tensor product rule of crystals.  $\square$

**Example 1.5.** We use the same notations as in Example 1.3. Consider the following diagram for  $\Psi_\lambda(T)$ .



Thus we have

$$i_T = 2, \quad \varepsilon_{i_T}(T) = 3 = \#\{\mu_3^{(1)}, \mu_4^{(1)}, \mu_3^{(2)}\},$$

and

$$\tilde{e}_2^3 T = T^+ = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 4 & 6 \\ \hline 2 & 2 & 4 & 5 & & \\ \hline 3 & 5 & & & & \\ \hline 5 & 6 & & & & \\ \hline 6 & & & & & \\ \hline \end{array}.$$

## 2. THE CRYSTAL $B(\infty)$ AND marginally large TABLEAUX

In this section, we recall the realization of the  $U_q(\mathfrak{sl}_{n+1})$ -crystal  $B(\infty)$  in terms of *marginally large* tableaux given in [4, 9, 20].

### Definition 2.1.

- (1) A semistandard tableau  $T \in \mathbf{B}(\lambda)$  is *large* if it consists of  $n$  non-empty rows, and if for each  $i = 1, \dots, n$ , the number of boxes having the entry  $i$  in the  $i$ th row is strictly greater than the number of all boxes in the  $(i+1)$ th row.
- (2) A large tableau  $T$  is *marginally large* if for each  $i = 1, \dots, n$ , the number of boxes having the entry  $i$  in the  $i$ th row is greater than the number of all boxes in the  $(i+1)$ th row by exactly one. In particular, the  $n$ th row of  $T$  should contain one box having the entry  $n$ .

We consider the following tableau:

$$T_0 := \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}.$$

For each marginally large tableau  $T$ , we construct a left-infinite extension of  $T$  obtained by adding infinitely many copies of  $T_0$  to the left of  $T$ . When there is no danger of confusion, we identify a marginally large tableau  $T$  with the left-infinite extension of  $T$ .

**Example 2.2.** Let  $\mathfrak{g} = \mathfrak{sl}_4$ . The following tableau  $T$  is marginally large:

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & & & & \\ \hline 3 & 4 & & & & & \\ \hline \end{array}.$$

The left-infinite extension of  $T$  obtained by adding infinitely many copies of  $T_0$  to the left of  $T$  is given as follows.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline \cdots & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ \hline \cdots & 2 & 2 & 2 & 2 & & & & \\ \hline \cdots & 3 & 3 & 4 & & & & & \\ \hline \end{array}.$$



Let  $\mathbf{B}(\infty)$  be the set of all left-infinite extensions of marginally large tableaux. The Kashiwara operators  $\tilde{f}_i, \tilde{e}_i$  ( $i \in I$ ) on  $\mathbf{B}(\infty)$  are defined as follows ([20]):

- (B1) We consider the infinite sequence of entries obtained by taking the Far-Eastern reading of  $T \in \mathbf{B}(\infty)$ . To each entry  $b$  in this sequence, we assign  $-$  if  $b = i + 1$  and  $+$  if  $b = i$ . Otherwise we put nothing. From this sequence of  $+$ 's and  $-$ 's, cancel out all  $(+, -)$  pairs. The remaining sequence is called the  $i$ -signature of  $T$ .
- (B2) Denote by  $T'$  the tableau obtained from  $T$  by replacing the entry  $i$  by  $i + 1$  corresponding to the leftmost  $+$  in the  $i$ -signature of  $T$ .
  - If  $T'$  is marginally large, then we define  $\tilde{f}_i T$  to be  $T'$ .
  - If  $T'$  is not marginally large, then define  $\tilde{f}_i T$  to be the marginally large tableau obtained by pushing all the rows appearing below the changed box in  $T'$  to the left by one box.
- (B3) Denote by  $T''$  the tableau obtained from  $T$  by replacing the entry  $i$  by  $i - 1$  corresponding to the rightmost  $-$  in the  $i$ -signature of  $T$ .
  - If  $T''$  is marginally large, then we define  $\tilde{e}_i T$  to be  $T''$ .
  - If  $T''$  is not marginally large, then define  $\tilde{e}_i T$  to be the marginally large tableau obtained by pushing all the rows appearing below the changed box in  $T''$  to the right by one box.
- (B4) If there is no  $-$  in the  $i$ -signature of  $T$ , we define  $\tilde{e}_i T = 0$ .

Let  $T$  be a marginally large tableau in  $\mathbf{B}(\infty)$ . For each  $i = 1, \dots, n$ , suppose that the  $i$ th row of  $T$  contains  $b_j^i$ -many  $j$ 's and infinitely many  $i$ 's. Define the maps  $\text{wt} : \mathbf{B}(\infty) \rightarrow P$ ,  $\varphi_i, \varepsilon_i : \mathbf{B}(\infty) \rightarrow \mathbb{Z}$  by

$$\begin{aligned} \text{wt}(T) &:= - \sum_{j=1}^n \left( \sum_{k=j+1}^{n+1} b_k^1 + \sum_{k=j+1}^{n+1} b_k^2 + \cdots + \sum_{k=j+1}^{n+1} b_k^j \right) \alpha_j, \\ \varepsilon_i(T) &:= \text{the number of } -\text{'s in the } i\text{-signature of } T, \\ \varphi_i(T) &:= \varepsilon_i(T) + \langle h_i, \text{wt}(T) \rangle. \end{aligned}$$

**Proposition 2.3.** [20, Theorem 4.8] *The sextuple  $(\mathbf{B}(\infty), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$  becomes a  $U_q(\mathfrak{sl}_{n+1})$ -crystal, which is isomorphic to the crystal  $B(\infty)$  of  $U_q^-(\mathfrak{sl}_{n+1})$ .*

Note that the highest weight vector  $T_\infty$  of  $\mathbf{B}(\infty)$  is given as follows:

$$T_\infty = \begin{array}{|c|c|c|c|c|} \hline \cdots & 1 & 1 & 1 & 1 \\ \hline \cdots & 2 & 2 & 2 & \\ \hline \vdots & \vdots & \vdots & & \\ \hline \cdots & n & & & \\ \hline \end{array}$$

It was shown in [13] that there is a unique strict crystal embedding

$$(2.1) \quad \iota_\lambda : \mathbf{B}(\lambda) \hookrightarrow \mathbf{B}(\infty) \otimes \mathbf{T}^\lambda \otimes \mathbf{C} \quad \text{given by} \quad T_\lambda \mapsto T_\infty \otimes t_\lambda \otimes c,$$

where  $T_\lambda$  is the highest weight vector of  $\mathbf{B}(\lambda)$ . We now describe this crystal embedding explicitly. Let  $T$  be a semistandard tableau of  $\mathbf{B}(\lambda)$ . We consider the left-infinite extension  $T'$  obtained from  $T$  by adding infinitely many copies of  $T_0$  to the left of  $T$ . Then we construct the marginally large tableau

$T_{\text{ml}}$  from  $T'$  by shifting the rows of  $T'$  in an appropriate way. Note that  $T_{\text{ml}}$  is uniquely determined. For example, if

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & & \\ \hline 4 & & & & \\ \hline \end{array},$$

then we have

$$T_{\text{ml}} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline \cdots & 2 & 2 & 2 & 3 & 3 & & & & \\ \hline \cdots & 3 & 4 & & & & & & & \\ \hline \end{array}.$$

Now the crystal embedding  $\iota_\lambda : \mathbf{B}(\lambda) \hookrightarrow \mathbf{B}(\infty) \otimes \mathbf{T}^\lambda \otimes \mathbf{C}$  is given by  $T \mapsto T_{\text{ml}} \otimes t_\lambda \otimes c$  [20].

For  $T \in \mathbf{B}(\infty)$ , we denote by  $a_{ij}^T$  the entry in the  $j$ th box from the right in the  $i$ th row of  $T$ . Define a map  $\Psi_\infty : \mathbf{B}(\infty) \rightarrow \mathcal{Y}^n$  by

$$(2.2) \quad \Psi_\infty(T) := (\mu^{(1)}, \dots, \mu^{(n)}),$$

where  $\mu^{(k)} = (a_{k,1}^T - k, a_{k,2}^T - k, \dots)$  for  $k = 1, \dots, n$ . Since

$$a_{k,j}^T - k = 0 \quad \text{for } j \gg 0,$$

the Young diagram  $\mu^{(k)}$  is well-defined for each  $k$ . Then, by construction, for any  $T \in \mathbf{B}(\lambda)$ , we have

$$\Psi_\lambda(T) = \Psi_\infty(T_{\text{ml}})$$

up to adding the empty Young diagrams. More precisely, we have the following lemma.

**Lemma 2.4.** *Let  $T$  be a semistandard tableau of  $\mathbf{B}(\lambda)$ , and  $\iota_\lambda(T) = T_{\text{ml}} \otimes t_\lambda \otimes c$ . Then  $\Psi_\infty(T_{\text{ml}})$  is the  $n$ -tuple of Young diagrams obtained from  $\Psi_\lambda(T)$  by adding the empty Young diagrams.*

### 3. KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF TYPE $A_n$

In this section, we review the basic properties of Khovanov-Lauda-Rouquier algebras [15, 16, 19, 21]. Let  $\alpha, \beta \in Q^+$  and  $d = \text{ht}(\alpha), d' = \text{ht}(\beta)$ . Define

$$I^\alpha := \{\mathbf{i} = (i_1, \dots, i_d) \in I^d \mid \alpha_{i_1} + \dots + \alpha_{i_d} = \alpha\}.$$

Then the symmetric group  $\Sigma_d$  acts on  $I^\alpha$  naturally. Let  $\Sigma_{d+d'}/\Sigma_d \times \Sigma_{d'}$  be the set of the minimal length coset representatives of  $\Sigma_d \times \Sigma_{d'}$  in  $\Sigma_{d+d'}$ . The following proposition is well-known.

**Proposition 3.1.** [5, Chapter 2.1], [19, Section 2.2]

*There is a 1-1- correspondence between  $\Sigma_{d+d'}/\Sigma_d \times \Sigma_{d'}$  and the set of all shuffles of  $\mathbf{i}$  and  $\mathbf{j}$ , where  $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$  and  $\mathbf{j} = (j_1, \dots, j_{d'}) \in I^\beta$ .*

For  $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$  and  $\mathbf{j} = (j_1, \dots, j_{d'}) \in I^\beta$ , we denote by  $\mathbf{i} * \mathbf{j}$  the concatenation of  $\mathbf{i}$  and  $\mathbf{j}$ :

$$\mathbf{i} * \mathbf{j} := (i_1, \dots, i_d, j_1, \dots, j_{d'}) \in I^{\alpha+\beta}.$$

**Definition 3.2.** Let  $\alpha \in Q^+$  and  $d = \text{ht}(\alpha)$ . The *Khovanov-Lauda-Rouquier algebra*  $R(\alpha)$  of type  $A_n$  corresponding to  $\alpha \in Q^+$  is the associative graded  $\mathbb{C}$ -algebra generated by  $1_{\mathbf{i}}$  ( $\mathbf{i} \in I^\alpha$ ),  $x_k$  ( $1 \leq k \leq d$ ),  $\tau_t$  ( $1 \leq t \leq d-1$ ) with the following defining relations :

$$\begin{aligned}
 (3.1) \quad & 1_{\mathbf{i}} 1_{\mathbf{j}} = \delta_{\mathbf{ij}} 1_{\mathbf{i}}, \quad x_k 1_{\mathbf{i}} = 1_{\mathbf{i}} x_k, \quad \tau_t 1_{\mathbf{i}} = 1_{\tau_t(\mathbf{i})} \tau_t, \\
 & x_k x_l = x_l x_k, \\
 & \tau_t \tau_s = \tau_s \tau_t \text{ if } |t - s| > 1, \\
 & \tau_t \tau_t 1_{\mathbf{i}} = \begin{cases} 0 & \text{if } i_t = i_{t+1}, \\ 1_{\mathbf{i}} & \text{if } |i_t - i_{t+1}| > 1, \\ (x_t + x_{t+1}) 1_{\mathbf{i}} & \text{if } |i_t - i_{t+1}| = 1, \end{cases} \\
 & (\tau_t \tau_{t+1} \tau_t - \tau_{t+1} \tau_t \tau_{t+1}) 1_{\mathbf{i}} = \begin{cases} 1_{\mathbf{i}} & \text{if } i_t = i_{t+2} \text{ and } |i_t - i_{t+1}| = 1, \\ 0 & \text{otherwise,} \end{cases} \\
 & (\tau_t x_k - x_{\tau_t(k)} \tau_t) 1_{\mathbf{i}} = \begin{cases} 1_{\mathbf{i}} & \text{if } k = t \text{ and } i_t = i_{t+1}, \\ -1_{\mathbf{i}} & \text{if } k = t+1 \text{ and } i_t = i_{t+1}, \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

For simplicity, we set  $R(0) = \mathbb{C}$ . The grading on  $R(\alpha)$  is given by

$$\deg(1_{\mathbf{i}}) = 0, \quad \deg(x_k 1_{\mathbf{i}}) = 2, \quad \deg(\tau_t 1_{\mathbf{i}}) = -a_{i_t, i_{t+1}}.$$

For  $\lambda = \sum_{i=1}^n a_i \varpi_i \in P^+$ , let  $I^\lambda(\alpha)$  be the two-side ideal of  $R(\alpha)$  generated by  $x_1^{a_{i_1}} 1_{\mathbf{i}}$  ( $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$ ), and define

$$R^\lambda(\alpha) := R(\alpha) / I^\lambda(\alpha).$$

The algebra  $R^\lambda(\alpha)$  is called the *cyclotomic quotient* of  $R(\alpha)$  at  $\lambda$ .

Let  $R(\alpha)\text{-fmod}$  (resp.  $R^\lambda(\alpha)\text{-fmod}$ ) be the category of finite dimensional graded  $R(\alpha)$ -modules (resp.  $R^\lambda(\alpha)$ -modules). For any irreducible graded module  $M \in R^\lambda(\alpha)\text{-fmod}$ ,  $M$  can be viewed as an irreducible graded  $R(\alpha)$ -module annihilated by  $I^\lambda(\alpha)$ , which defines a functor

$$\text{infl}_\lambda : R^\lambda(\alpha)\text{-fmod} \rightarrow R(\alpha)\text{-fmod}.$$

For  $M \in R^\lambda(\alpha)\text{-fmod}$ ,  $\text{infl}_\lambda M$  is called the *inflation* of  $M$ . On the other hand, from the natural projection  $R(\alpha) \rightarrow R^\lambda(\alpha)$ , we define the functor  $\text{pr}_\lambda : R(\alpha)\text{-fmod} \rightarrow R^\lambda(\alpha)\text{-fmod}$  by

$$\text{pr}_\lambda N := N / I^\lambda(\alpha) N \quad \text{for } N \in R(\alpha)\text{-fmod}.$$

From now on, when there is no danger of confusion, we identify any irreducible graded  $R^\lambda(\alpha)$ -module with an irreducible graded  $R(\alpha)$ -module annihilated by  $I^\lambda(\alpha)$  via the functor  $\text{infl}_\lambda$ .

The algebra  $R(\alpha)$  has a graded anti-involution

$$(3.2) \quad \psi : R(\alpha) \longrightarrow R(\alpha)$$

which is the identity on generators. Using this anti-involution, for any finite dimensional graded  $R(\alpha)$ -module  $M$ , the dual space  $M^* := \text{Hom}_{\mathbb{C}}(M, \mathbb{C})$  of  $M$  has the  $R(\alpha)$ -module structure given by

$$(r \cdot f)(m) := f(\psi(r)m) \quad (r \in R(\alpha), m \in M).$$

Note that, if  $M$  is irreducible, then  $M \simeq M^*$  by [16, Theorem 3.17].

Given  $M = \bigoplus_{i \in \mathbb{Z}} M_i$ , let  $M\langle k \rangle$  denote the graded module obtained from  $M$  by shifting the grading by  $k$ ; i.e.,

$$M\langle k \rangle := \bigoplus_{i \in \mathbb{Z}} M\langle k \rangle_i,$$

where  $M\langle k \rangle_i := M_{i-k}$  for  $i \in \mathbb{Z}$ . We define the  $q$ -dimension  $\text{qdim}(M)$  of  $M = \bigoplus_{i \in \mathbb{Z}} M_i$  to be

$$\text{qdim}(M) := \sum_{i \in \mathbb{Z}} (\dim M_i) q^i.$$

Set

$$\begin{aligned} R &:= \bigoplus_{\alpha \in Q^+} R(\alpha), & K_0(R) &:= \bigoplus_{\alpha \in Q^+} K_0(R(\alpha)\text{-fmod}), \\ R^\lambda &:= \bigoplus_{\alpha \in Q^+} R^\lambda(\alpha), & K_0(R^\lambda) &:= \bigoplus_{\alpha \in Q^+} K_0(R^\lambda(\alpha)\text{-fmod}), \end{aligned}$$

where  $K_0(R(\alpha)\text{-fmod})$  (resp.  $K_0(R^\lambda(\alpha)\text{-fmod})$ ) is the Grothendieck group of  $R(\alpha)\text{-fmod}$  (resp.  $R^\lambda(\alpha)\text{-fmod}$ ). For  $M \in R(\alpha)\text{-fmod}$  (resp.  $R^\lambda(\alpha)\text{-fmod}$ ), we denote by  $[M]$  the isomorphism class of  $M$  in  $K_0(R(\alpha)\text{-fmod})$  (resp.  $K_0(R^\lambda(\alpha)\text{-fmod})$ ). Then  $K_0(R)$  (resp.  $K_0(R^\lambda)$ ) has the  $\mathbb{Z}[q, q^{-1}]$ -module structure given by  $q[M] = [M\langle 1 \rangle]$ .

Define the  $q$ -character  $\text{ch}_q(M)$  (resp. *character*  $\text{ch}(M)$ ) of  $M \in R(\alpha)\text{-fmod}$  by

$$\text{ch}_q(M) := \sum_{\mathbf{i} \in I^\alpha} \text{qdim}(1_{\mathbf{i}} M) \mathbf{i} \quad (\text{resp. } \text{ch}(M) := \sum_{\mathbf{i} \in I^\alpha} \dim(1_{\mathbf{i}} M) \mathbf{i}).$$

Note that the evaluation of  $\text{qdim}(1_{\mathbf{i}} M)$  at  $q = 1$  is  $\dim(1_{\mathbf{i}} M)$ . For  $M \in R(\alpha)\text{-fmod}$  and  $N \in R(\beta)\text{-fmod}$ , we set

$$\begin{aligned} \text{ch}_q(M) * \text{ch}_q(N) &:= \sum_{\mathbf{i} \in I^\alpha, \mathbf{j} \in I^\beta} \text{qdim}(1_{\mathbf{i}} M) \text{qdim}(1_{\mathbf{j}} N) \mathbf{i} * \mathbf{j}, \\ \text{ch}(M) * \text{ch}(N) &:= \sum_{\mathbf{i} \in I^\alpha, \mathbf{j} \in I^\beta} \dim(1_{\mathbf{i}} M) \dim(1_{\mathbf{j}} N) \mathbf{i} * \mathbf{j}. \end{aligned}$$

For  $M, N \in R(\alpha)\text{-fmod}$ , let  $\text{Hom}(M, N)$  be the  $\mathbb{C}$ -vector space of degree preserving homomorphisms, and  $\text{Hom}(M\langle k \rangle, N) = \text{Hom}(M, N\langle -k \rangle)$  be the  $\mathbb{C}$ -vector space of homogeneous homomorphisms of degree  $k$ . Define

$$\text{HOM}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M, N\langle k \rangle).$$

Let  $\beta_1, \dots, \beta_k \in Q^+$  and set  $\beta = \beta_1 + \dots + \beta_k$ . Then there is a natural embedding

$$\iota_{\beta_1, \dots, \beta_k} : R(\beta_1) \otimes \dots \otimes R(\beta_k) \hookrightarrow R(\beta),$$

which yields the following functors from  $R(\beta_1) \otimes \dots \otimes R(\beta_k)\text{-fmod}$  to  $R(\beta)\text{-fmod}$ :

$$\begin{aligned} \text{Ind}_{\beta_1, \dots, \beta_k} - &:= R(\beta) \otimes_{R(\beta_1) \otimes \dots \otimes R(\beta_k)} -, \\ \text{coInd}_{\beta_1, \dots, \beta_k} - &:= \text{HOM}_{R(\beta_1) \otimes \dots \otimes R(\beta_k)}(R(\beta), -). \end{aligned}$$

The properties of the functors  $\text{Ind}$ ,  $\text{coInd}$  and  $\text{Res}$  are summarized in the following lemmas.

**Lemma 3.3.** [19, (2.3)] *For  $M \in R(\beta_1) \otimes \cdots \otimes R(\beta_k)$ -fmod and  $N \in R(\beta)$ -fmod, we have*

$$\begin{aligned} \mathrm{HOM}_{R(\beta)}(\mathrm{Ind}_{\beta_1, \dots, \beta_k} M, N) &\cong \mathrm{HOM}_{R(\beta_1) \otimes \cdots \otimes R(\beta_k)}(M, \mathrm{Res}_{\beta_1, \dots, \beta_k} N), \\ \mathrm{HOM}_{R(\beta)}(N, \mathrm{coInd}_{\beta_1, \dots, \beta_k} M) &\cong \mathrm{HOM}_{R(\beta_1) \otimes \cdots \otimes R(\beta_k)}(\mathrm{Res}_{\beta_1, \dots, \beta_k} N, M). \end{aligned}$$

**Theorem 3.4.** [19, Theorem 2.2] *Let  $M_i \in R(\beta_i)$ -fmod ( $i = 1, \dots, k$ ) and*

$$K := - \sum_{i>j} (\beta_i | \beta_j),$$

where  $(\cdot | \cdot)$  is the nondegenerate symmetric bilinear form on  $Q$  defined by  $(\alpha_i | \alpha_j) = a_{ij}$  ( $i, j \in I$ ). Then there exists a homogeneous isomorphism

$$\mathrm{Ind}_{\beta_1, \dots, \beta_k} M_1 \boxtimes \cdots \boxtimes M_k \cong \mathrm{coInd}_{\beta_k, \dots, \beta_1} (M_k \boxtimes \cdots \boxtimes M_1) \langle K \rangle.$$

When there is no ambiguity, we will write  $\mathrm{Res}$ ,  $\mathrm{Ind}$ ,  $\mathrm{coInd}$  for  $\mathrm{Res}_{\beta_1, \dots, \beta_k}$ ,  $\mathrm{Ind}_{\beta_1, \dots, \beta_k}$  and  $\mathrm{coInd}_{\beta_1, \dots, \beta_k}$ , respectively.

We first consider the special case when  $\alpha = m\alpha_i$ . It is known that  $R(m\alpha_i)$  is isomorphic to the nilHecke ring  $NH_m$  (see [16, Example 2.2]). Thus  $R(m\alpha_i)$  has only one irreducible representation

$$(3.3) \quad L(i^m) \cong \mathrm{Ind}_{\mathbb{C}[x_1, \dots, x_m]} \mathbf{1}$$

up to grading shift, where  $\mathbf{1}$  is the 1-dimensional trivial module over  $\mathbb{C}[x_1, \dots, x_m]$ . We define  $\mathrm{ch}_q(\mathbf{1}) := (i, \dots, i)$ . Since  $\dim L(i^m) = m!$ , for any  $M \in R(\alpha)$ -fmod and  $\mathbf{i} = (\cdots, \underbrace{i, \dots, i}_m, \cdots) \in I^\alpha$  with  $\dim(\mathbf{1}_{\mathbf{i}} M) > 0$ , we have

$$(3.4) \quad \dim(\mathbf{1}_{\mathbf{i}} M) \geq m!.$$

Take a nonzero element  $\zeta$  in  $\mathbf{1}$ . Then  $L(i^m)$  is generated by  $1 \otimes \zeta$  and, by [16, Theorem 2.5],  $L(i^m)$  has a basis  $\{w \cdot 1 \otimes \zeta \mid w \in \Sigma_m\}$ . Set  $L^0 = \{0\}$  and

$$L^k := \{v \in L(i^m) \mid x_m^k \cdot v = 0\} \quad (k = 1, 2, \dots, m).$$

Since  $x_m$  commutes with all  $x_i$  ( $i = 1, \dots, m-1$ ) and  $\tau_j$  ( $j = 1, \dots, m-2$ ),  $L^k$  can be considered as  $R((m-1)\alpha_i)$ -module. Moreover, by a direct computation, we have

$$(3.5) \quad L^k = \{w\tau_{m-1} \cdots \tau_{m-k+1} \cdot 1 \otimes \zeta \mid w \in \Sigma_{m-1}\}.$$

It follows that  $L^k/L^{k-1}$  is isomorphic to  $L(i^{m-1})$  for each  $k = 1, \dots, m$ .

We now return to the general case. Let  $M$  be a finite dimensional graded  $R(\alpha)$ -module. For any  $\beta \in Q^+$ , set  $1_\beta := \sum_{\mathbf{i} \in I^\beta} \mathbf{1}_{\mathbf{i}}$ . For  $i \in I$ , define

$$(3.6) \quad \begin{aligned} \Delta_i M &:= 1_{\alpha - k\alpha_i} \otimes 1_{k\alpha_i} M, \\ e_i M &:= \mathrm{Res}_{\alpha - \alpha_i}^{\alpha - \alpha_i, \alpha_i} \circ \Delta_i M. \end{aligned}$$

Then  $e_i$  may be considered as a functor:  $K_0(R(\alpha)\text{-fmod}) \rightarrow K_0(R(\alpha - \alpha_i)\text{-fmod})$ .

**Lemma 3.5.** *Let  $M \in R(\alpha)$ -fmod and  $N \in R(\beta)$ -fmod. Then we have the following exact sequence:*

$$0 \rightarrow \mathrm{Ind}_{\alpha, \beta - \alpha_i} M \boxtimes e_i N \rightarrow e_i(\mathrm{Ind}_{\alpha, \beta} M \boxtimes N) \rightarrow \mathrm{Ind}_{\alpha - \alpha_i, \beta} e_i M \boxtimes N \langle -\beta(h_i) \rangle \rightarrow 0.$$

*Proof.* Our assertion follows from the Khovanov-Lauda-Rouquier algebra version of the Mackey's theorem [16, Proposition 2.18].  $\square$

**Proposition 3.6.** [16, Corollary 2.15] *For any finitely-generated graded  $R(\alpha)$ -module  $M$ , we have*

$$\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(1-a_{ij}-r)} e_j e_i^{(r)} [M] = 0,$$

where  $e_i^{(r)} [M] = \frac{1}{[r]_q!} [e_i^r M]$  for  $i \in I$  and  $r \in \mathbb{Z}_{\geq 0}$ .

Let us reinterpret the quantum Serre relations given in Proposition 3.6. Let  $M$  be a finite-dimensional graded  $R(\alpha)$ -module. Consider the sequences  $\mathbf{i}_{\{i,j\}}, \mathbf{i}_{\{j,i\}} \in I^\alpha$  of the form:

$$\mathbf{i}_{\{i,j\}} := \mathbf{k}_1 * (i, j) * \mathbf{k}_2, \quad \mathbf{i}_{\{j,i\}} := \mathbf{k}_1 * (j, i) * \mathbf{k}_2,$$

where  $\mathbf{k}_1, \mathbf{k}_2$  are sequences satisfying  $\mathbf{k}_1 * \mathbf{k}_2 \in I^{\alpha-\alpha_i-\alpha_j}$ . Suppose  $|i-j| > 1$ . It follows from Proposition 3.6 that

$$(3.7) \quad \dim(1_{\mathbf{i}_{\{i,j\}}} M) = \dim(1_{\mathbf{i}_{\{j,i\}}} M).$$

We now consider the case  $|i-j| = 1$ . Let

$$\begin{aligned} \mathbf{i}_{\{i\pm 1, i, i\}} &:= \mathbf{k}_1 * (i \pm 1, i, i) * \mathbf{k}_2, \\ \mathbf{i}_{\{i, i\pm 1, i\}} &:= \mathbf{k}_1 * (i, i \pm 1, i) * \mathbf{k}_2, \\ \mathbf{i}_{\{i, i, i\pm 1\}} &:= \mathbf{k}_1 * (i, i, i \pm 1) * \mathbf{k}_2 \end{aligned}$$

for some sequences  $\mathbf{k}_1, \mathbf{k}_2$  with  $\mathbf{k}_1 * \mathbf{k}_2 \in I^{\alpha-2\alpha_i-\alpha_{i\pm 1}}$ . Then, from Proposition 3.6, we have

$$(3.8) \quad 2 \dim(1_{\mathbf{i}_{\{i, i\pm 1, i\}}} M) = \dim(1_{\mathbf{i}_{\{i\pm 1, i, i\}}} M) + \dim(1_{\mathbf{i}_{\{i, i, i\pm 1\}}} M).$$

Let  $\mathfrak{B}(\infty)$  denote the set of isomorphism classes of irreducible graded  $R$ -modules, and define

$$\begin{aligned} \text{wt}(M) &:= -\alpha, \\ \tilde{e}_i M &:= \text{soc } e_i M, \\ \tilde{f}_i M &:= \text{hd } \text{Ind}_{\alpha, \alpha_i} M \boxtimes L(i), \\ \varepsilon_i(M) &:= \max\{k \geq 0 \mid \tilde{e}_i^k M \neq 0\} \\ \varphi_i(M) &:= \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle. \end{aligned}$$

**Theorem 3.7.** [19, Theorem 7.4] *The sextuple  $(\mathfrak{B}(\infty), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$  becomes a crystal, which is isomorphic to the crystal  $B(\infty)$  of  $U_q^-(\mathfrak{sl}_{n+1})$ .*

For  $M \in R^\lambda(\alpha)\text{-fmod}$  and  $N \in R(\alpha)\text{-fmod}$ , let  $\text{infl}_\lambda M$  be the inflation of  $M$ , and  $\text{pr}_\lambda N$  be the quotient of  $N$  by  $I^\lambda(\alpha)N$ . Let  $\mathfrak{B}(\lambda)$  denote the set of isomorphism classes of irreducible  $R^\lambda$ -modules,

and for  $M \in R^\lambda(\alpha)\text{-fmod}$ , define

$$\begin{aligned} \text{wt}^\lambda(M) &:= \lambda - \alpha, \\ \tilde{e}_i^\lambda M &:= \text{pr}_\lambda \circ \tilde{e}_i \circ \text{infl}_\lambda M, \\ \tilde{f}_i^\lambda M &:= \text{pr}_\lambda \circ \tilde{f}_i \circ \text{infl}_\lambda M, \\ \varepsilon_i^\lambda(M) &:= \max\{k \geq 0 \mid (\tilde{e}_i^\lambda)^k M \neq 0\}, \\ \varphi_i^\lambda(M) &:= \varepsilon_i^\lambda(M) + \langle h_i, \text{wt}^\lambda(M) \rangle. \end{aligned}$$

**Theorem 3.8.** [19, Theorem 7.5] *The sextuple  $(\mathfrak{B}(\lambda), \text{wt}^\lambda, \tilde{e}_i^\lambda, \tilde{f}_i^\lambda, \varepsilon_i^\lambda, \varphi_i^\lambda)$  becomes a crystal, which is isomorphic to the crystal  $B(\lambda)$  of the irreducible highest weight  $U_q(\mathfrak{sl}_{n+1})$ -module  $V(\lambda)$ .*

The following lemma is an analogue of [17, Theorem 5.5.1].

**Lemma 3.9.** *Let  $M$  be an irreducible  $R(\alpha)$ -module. Set  $\varepsilon := \varepsilon_i(M)$ . Then we have*

(1)

$$[e_i M] = q^{-\varepsilon+1} [\varepsilon]_q [\tilde{e}_i M] + \sum_k c_k [N_k],$$

where  $N_k$  are irreducible modules with  $\varepsilon_i(N_k) < \varepsilon_i(\tilde{e}_i M) = \varepsilon - 1$ ,

(2)

$$[e_i^\varepsilon M] = q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\tilde{e}_i^\varepsilon M].$$

*Proof.* Since the assertion (2) follows from the assertion (1) immediately, it suffices to prove (1). By in [16, Lemma 3.8],

$$\Delta_{i^\varepsilon} M \cong N \boxtimes L(i^\varepsilon)$$

for some irreducible  $N \in R(\alpha - \varepsilon\alpha_i)\text{-fmod}$  with  $\varepsilon_i(N) = 0$ . Then we have

$$N \boxtimes L(i^\varepsilon) \xrightarrow{\sim} \Delta_{i^\varepsilon} M \subset \text{Res}_{\alpha - \varepsilon\alpha_i, \varepsilon\alpha_i} M,$$

which yields

$$0 \longrightarrow K \longrightarrow \text{Ind}_{\alpha - \varepsilon\alpha_i, \varepsilon\alpha_i} N \boxtimes L(i^\varepsilon) \longrightarrow M \longrightarrow 0$$

for some  $R(\alpha)$ -module  $K$ . Note that  $\varepsilon_i(K) < \varepsilon$ .

On the other hand, it follows from (3.3) and (3.5) that

$$[\Delta_i L(i^\varepsilon)] = q^{-\varepsilon+1} [\varepsilon]_q [L(i^{\varepsilon-1}) \boxtimes L(i)].$$

Since  $\varepsilon_i(N) = 0$ , it follows from [16, Proposition 2.18] that

$$[\Delta_i \text{Ind}_{\alpha - \varepsilon\alpha_i, \varepsilon\alpha_i} N \boxtimes L(i^\varepsilon)] = q^{-\varepsilon+1} [\varepsilon]_q [\text{Ind}_{\alpha - \varepsilon\alpha_i, (\varepsilon-1)\alpha_i, \alpha_i}^{\alpha - \alpha_i, \alpha_i} N \boxtimes L(i^{\varepsilon-1}) \boxtimes L(i)].$$

By [16, Lemma 3.9] and [16, Lemma 3.13], we obtain

$$\text{hd}(\text{Ind}_{\alpha - \varepsilon\alpha_i, (\varepsilon-1)\alpha_i, \alpha_i}^{\alpha - \alpha_i, \alpha_i} N \boxtimes L(i^{\varepsilon-1}) \boxtimes L(i)) \cong (\tilde{f}_i^{\varepsilon-1} N) \boxtimes L(i) \cong \tilde{e}_i M \boxtimes L(i),$$

and all the other composition factors of  $\text{Ind}_{\alpha-\varepsilon\alpha_i, (\varepsilon-1)\alpha_i, \alpha_i}^{\alpha-\alpha_i, \alpha_i} N \boxtimes L(i^{\varepsilon-1}) \boxtimes L(i)$  are of the form  $L \boxtimes L(i)$  with  $\varepsilon_i(L) < \varepsilon - 1$ . Moreover, since  $\varepsilon_i(K) < \varepsilon$ , all composition factors of  $\Delta_i(K)$  are of the form  $L \boxtimes L(i)$  with  $\varepsilon_i(L) < \varepsilon - 1$ . Therefore, we obtain

$$[e_i M] = q^{-\varepsilon+1} [\varepsilon]_q [\tilde{e}_i M] + \sum_k c_k [N_k],$$

where  $N_k$  are irreducible modules with  $\varepsilon_i(N_k) < \varepsilon_i(\tilde{e}_i M) = \varepsilon - 1$ .  $\square$

The following lemmas are analogues of [22, Proposition 8, Proposition 9], which will play crucial roles in proving our main theorem.

**Lemma 3.10.** *For  $\beta_1, \dots, \beta_k \in Q^+$ , let  $\gamma_i$  be a 1-dimensional graded  $R(\beta_i)$ -module.*

(1) *If  $Q$  is any graded quotient of  $\text{Ind}_{\beta_1, \dots, \beta_k} \gamma_1 \boxtimes \dots \boxtimes \gamma_k$ , then  $\text{ch} Q$  contains*

$$\text{ch}(\gamma_1) * \dots * \text{ch}(\gamma_k).$$

(2) *If  $L$  is any graded submodule of  $\text{Ind}_{\beta_1, \dots, \beta_k} \gamma_1 \boxtimes \dots \boxtimes \gamma_k$ , then  $\text{ch} L$  contains*

$$\text{ch}(\gamma_k) * \dots * \text{ch}(\gamma_1).$$

*Proof.* It follows from Lemma 3.3 that

$$\text{HOM}_{R(\beta_1) \otimes \dots \otimes R(\beta_k)}(\gamma_1 \boxtimes \dots \boxtimes \gamma_k, \text{Res}_{\beta_1, \dots, \beta_k} Q)$$

is nontrivial, which implies that  $\text{ch} Q$  contains the concatenation  $\text{ch}(\gamma_1) * \dots * \text{ch}(\gamma_k)$ .

Consider now the assertion (2). By Lemma 3.3 and Theorem 3.4, we have

$$\begin{aligned} & \text{HOM}_{R(\beta_1 + \dots + \beta_k)}(L, \text{Ind}_{\beta_1, \dots, \beta_k} \gamma_1 \boxtimes \dots \boxtimes \gamma_k) \\ & \cong \text{HOM}_{R(\beta_1 + \dots + \beta_k)}(L, \text{coInd}_{\beta_k, \dots, \beta_1} \gamma_k \boxtimes \dots \boxtimes \gamma_1) \\ & \cong \text{HOM}_{R(\beta_k) \otimes \dots \otimes R(\beta_1)}(\text{Res}_{\beta_k, \dots, \beta_1} L, \gamma_k \boxtimes \dots \boxtimes \gamma_1). \end{aligned}$$

Since the above spaces are non-trivial, the assertion (2) follows.  $\square$

**Lemma 3.11.** *Let  $\beta_1, \dots, \beta_k \in Q^+$  and let  $M$  be an irreducible  $R(\beta_1) \otimes \dots \otimes R(\beta_k)$ -module. Assume that  $\mathbf{i} \in I^{\beta_1 + \dots + \beta_k}$  appears in  $\text{ch}(\text{Ind}_{\beta_1, \dots, \beta_k} M)$  with coefficient  $m$ .*

- (1) *Suppose that  $\mathbf{i}$  occurs with coefficient  $m$  in the character of any submodule of  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ . Then  $\text{socInd}_{\beta_1, \dots, \beta_k} M$  is irreducible and occurs with multiplicity one as a composition factor of  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ . If  $\mathbf{i}$  occurs in  $\text{ch}(\text{hdInd}_{\beta_1, \dots, \beta_k} M)$ , then  $\text{Ind}_{\beta_1, \dots, \beta_k} M$  is irreducible.*
- (2) *Suppose that  $\mathbf{i}$  occurs with coefficient  $m$  in the character of any quotient of  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ . Then  $\text{hdInd}_{\beta_1, \dots, \beta_k} M$  is irreducible and occurs with multiplicity one as a composition factor of  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ . If  $\mathbf{i}$  occurs in  $\text{ch}(\text{socInd}_{\beta_1, \dots, \beta_k} M)$ , then  $\text{Ind}_{\beta_1, \dots, \beta_k} M$  is irreducible.*

*Proof.* Let  $L$  be a component of  $\text{socInd}_{\beta_1, \dots, \beta_k} M$ . By hypothesis,  $\text{ch} L$  contains  $\mathbf{i}$  with coefficient  $m$  as a term. Since  $\mathbf{i}$  occurs with coefficient  $m$  in the character of any submodule,  $\text{socInd}_{\beta_1, \dots, \beta_k} M$  should be irreducible. In a similar manner, one can show that  $\text{socInd}_{\beta_1, \dots, \beta_k} M$  occurs with multiplicity one as a composition factor of  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ . Suppose that  $\mathbf{i}$  occurs in  $\text{ch}(\text{hdInd}_{\beta_1, \dots, \beta_k} M)$ . Since the



multiplicity of  $\mathbf{i}$  in  $\text{socInd}_{\beta_1, \dots, \beta_k} M$  is equal to the multiplicity of  $\mathbf{i}$  in  $\text{Ind}_{\beta_1, \dots, \beta_k} M$ ,  $\text{Ind}_{\beta_1, \dots, \beta_k} M$  should be irreducible.

The assertion (2) can be proved in a similar manner.  $\square$

#### 4. IRREDUCIBLE $R^\lambda$ -MODULES AND SEMISTANDARD TABLEAUX

In this section, we prove the main results of our paper. We give an explicit construction of irreducible graded  $R^\lambda$ -modules (Theorem 4.5) and show that there exists an explicit crystal isomorphism  $\Phi_\lambda : \mathbf{B}(\lambda) \rightarrow \mathfrak{B}(\lambda)$  (Theorem 4.8). From now on, isomorphisms of modules are allowed to be homogeneous.

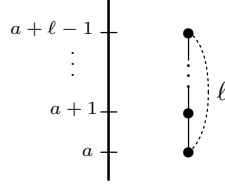
For  $a, \ell \in \mathbb{Z}_{>0}$  with  $a + \ell - 1 \leq n$ , let

$$\begin{aligned}\alpha_{(a; \ell)} &:= \alpha_a + \alpha_{a+1} + \dots + \alpha_{a+\ell-1} \in Q^+, \\ \mathbf{i}_{(a; \ell)} &:= (a, a+1, \dots, a+\ell-1) \in I^{\alpha_{(a; \ell)}}.\end{aligned}$$

Define  $\nabla_{(a; \ell)}$  to be the 1-dimensional  $R(\alpha_{(a; \ell)})$ -module  $\mathbb{C}v$  given by

$$(4.1) \quad x_i v = 0, \quad \tau_j v = 0, \quad 1_{\mathbf{i}} v = \begin{cases} v & \text{if } \mathbf{i} = \mathbf{i}_{(a; \ell)}, \\ 0 & \text{otherwise.} \end{cases}$$

The module  $\nabla_{(a; \ell)}$  can be visualized as follows:

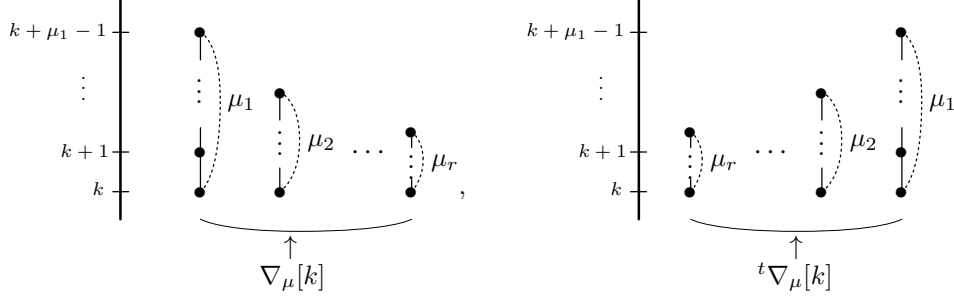


For simplicity, set  $\nabla_{(a; 0)} := \mathbb{C}$ . Note that  $\nabla_{(a; \ell)}$  is graded and  $\text{ch} \nabla_{(a; \ell)} = \mathbf{i}_{(a; \ell)}$ .

Let  $\mu = (\mu_1 \geq \dots \geq \mu_r > 0)$  be a Young diagram, and  $k \in \mathbb{Z}_{>0}$ . Suppose that  $k + \mu_1 - 1 \leq n$ . Define

$$(4.2) \quad \begin{aligned}\alpha_\mu[k] &:= \alpha_{(k; \mu_1)} + \dots + \alpha_{(k; \mu_r)} \in Q^+, \\ \nabla_\mu[k] &:= \nabla_{(k; \mu_1)} \boxtimes \nabla_{(k; \mu_2)} \boxtimes \dots \boxtimes \nabla_{(k; \mu_r)}, \\ {}^t \nabla_\mu[k] &:= \nabla_{(k; \mu_r)} \boxtimes \nabla_{(k; \mu_{r-1})} \boxtimes \dots \boxtimes \nabla_{(k; \mu_1)}.\end{aligned}$$

Pictorially, the modules  $\nabla_\mu[k]$  and  ${}^t \nabla_\mu[k]$  may be viewed as follows:



One of the key ingredients of the proof of Theorem 4.8 is the fact that  $\text{Ind}\nabla_\mu[k]$  is irreducible for any Young diagram  $\mu$  and  $k \in \mathbb{Z}_{>0}$ . To prove this, we need several lemmas. The following lemma may be obtained by translating the *linking rule* given in [22, Lemma 4] into the language of Khovanov-Lauda-Rouquier algebras.

**Lemma 4.1.** *Let  $a_i, \ell_i \in \mathbb{Z}_{>0}$  with  $a_i + \ell_i - 1 \leq n$  ( $i = 1, 2$ ).*

(1) *If  $a_1 + \ell_1 - 1 < a_2$ , then*

$$\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)} \cong \text{Ind}\nabla_{(a_2; \ell_2)} \boxtimes \nabla_{(a_1; \ell_1)},$$

*and  $\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)}$  is irreducible.*

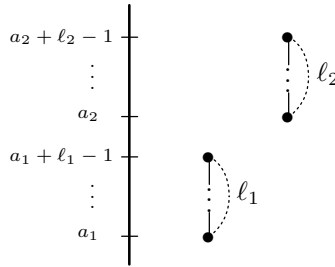
(2) *If  $a_2 \geq a_1$  and  $a_1 + \ell_1 \geq a_2 + \ell_2$ , then*

$$\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)} \cong \text{Ind}\nabla_{(a_2; \ell_2)} \boxtimes \nabla_{(a_1; \ell_1)},$$

*and  $\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)}$  is irreducible.*

*Proof.* Let  $\alpha := \alpha_{(a_1; \ell_1)} + \alpha_{(a_2; \ell_2)}$  and let  $\Sigma_{\ell_1 + \ell_2} / \Sigma_{\ell_1} \times \Sigma_{\ell_2}$  be the set of the minimal length coset representatives of  $\Sigma_{\ell_1} \times \Sigma_{\ell_2}$  in  $\Sigma_{\ell_1 + \ell_2}$ .

(1) The condition  $a_1 + \ell_1 - 1 < a_2$  can be visualized as follows.



By [16, Proposition 2.18], we have

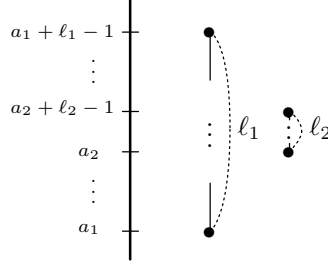
$$\text{ch}(\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)}) = \sum_{w \in \Sigma_{\ell_1 + \ell_2} / \Sigma_{\ell_1} \times \Sigma_{\ell_2}} w \cdot (\mathbf{i}_{(a_1; \ell_1)} * \mathbf{i}_{(a_2; \ell_2)}).$$

Note that each term in  $\text{ch}(\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)})$  has multiplicity 1. Let  $Q$  be a quotient of  $\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)}$ . It follows from Lemma 3.10 that  $\text{ch}(Q)$  contains  $\mathbf{i}_{(a_1; \ell_1)} * \mathbf{i}_{(a_2; \ell_2)}$  as a term. By (3.7), all terms in  $\text{ch}(\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)})$  occur in  $\text{ch}(Q)$ . Therefore,  $\text{Ind}\nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)}$  is irreducible. In the

same manner, one can prove that  $\text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)}$  is irreducible. Comparing the characters  $\text{ch}(\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)})$  and  $\text{ch}(\text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)})$ , by [16, Theorem 3.17], we conclude

$$\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)} \cong \text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)}.$$

(2) The conditions  $a_2 \geq a_1$  and  $a_1 + \ell_1 \geq a_2 + \ell_2$  can be visualized as follows.



Let

$$\mathbf{k} := (a_1, a_1 + 1, \dots, a_2, a_2, a_2 + 1, a_2 + 1, \dots, a_2 + \ell_2 - 1, a_2 + \ell_2 - 1, \dots, a_1 + \ell_1 - 1) \in I^\alpha.$$

By Proposition 3.1 and the identity

$$\text{ch}(\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)}) = \sum_{w \in \Sigma_{\ell_1+\ell_2}/\Sigma_{\ell_1} \times \Sigma_{\ell_2}} w \cdot (\mathbf{i}_{(a_1;\ell_1)} * \mathbf{i}_{(a_2;\ell_2)}),$$

it is easy to see that  $\mathbf{k}$  occurs in  $\text{ch}(\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)})$  with multiplicity  $2^{\ell_2}$ . On the other hand, by Lemma 3.10, for any quotient  $Q$  of  $\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)}$ ,  $\text{ch}(Q)$  contains  $\mathbf{i}_{(a_1;\ell_1)} * \mathbf{i}_{(a_2;\ell_2)}$  as a term. By (3.7),  $\text{ch}(Q)$  must have the following term

$$(a_1, \dots, a_2, a_2 + 1, a_2, a_2 + 2, \dots, a_1 + \ell_1 - 1, a_2 + 1, \dots, a_2 + \ell_2 - 1).$$

Hence by (3.8) and Proposition 3.1,  $\text{ch}(Q)$  contains

$$(a_1, \dots, a_2, a_2, a_2 + 1, a_2 + 2, \dots, a_1 + \ell_1 - 1, a_2 + 1, \dots, a_2 + \ell_2 - 1).$$

Continuing this process repeatedly,  $\text{ch}(Q)$  must contain the term  $\mathbf{k}$ . By (3.4), we deduce that  $\mathbf{k}$  occurs in  $\text{ch}(Q)$  with multiplicity  $2^{\ell_2}$ . In the same manner, for any submodule  $L$  of  $\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)}$ ,  $\text{ch}(L)$  contains  $\mathbf{k}$  with multiplicity  $2^{\ell_2}$ . Therefore, by Lemma 3.11, we conclude that  $\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)}$  is irreducible.

Similarly, one can prove that  $\text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)}$  is irreducible. Comparing the characters of  $\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)}$  and that of  $\text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)}$ , by [16, Theorem 3.17], we obtain

$$\text{Ind}\nabla_{(a_1;\ell_1)} \boxtimes \nabla_{(a_2;\ell_2)} \cong \text{Ind}\nabla_{(a_2;\ell_2)} \boxtimes \nabla_{(a_1;\ell_1)}.$$

□

**Lemma 4.2.**

(1) For  $a \in \mathbb{Z}_{>0}$  and  $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k > 0$  with  $a + \ell_1 - 1 \leq n$ , we have

$$\text{Ind}\nabla_{(a;\ell_1)} \boxtimes \dots \boxtimes \nabla_{(a;\ell_k)} \cong (\text{Ind}\nabla_{(a;\ell_k)} \boxtimes \dots \boxtimes \nabla_{(a;\ell_1)})^*.$$

(2) Let  $a_1, \dots, a_k \in \mathbb{Z}_{>0}$  and  $\ell_1 \geq \ell_2 \geq \dots \geq \ell_k > 0$ . If

$$a_i + \ell_i - 1 = a_j + \ell_j - 1 \leq n \quad (i \neq j),$$

then we have

$$\text{Ind} \nabla_{(a_1; \ell_1)} \boxtimes \dots \boxtimes \nabla_{(a_k; \ell_k)} \cong (\text{Ind} \nabla_{(a_k; \ell_k)} \boxtimes \dots \boxtimes \nabla_{(a_1; \ell_1)})^*.$$

*Proof.* We first prove the assertion (1). Let

$$\nabla_i := \nabla_{(a; \ell_i)}, \quad \beta_i := \alpha_{(a; \ell_i)} \quad \text{for } i = 1, \dots, k,$$

and  $\beta := \sum_{i=1}^k \beta_i$ . Take a nonzero element  $v_i \in \nabla_i$  for each  $i = 1, \dots, k$ . From Lemma 3.3 and Theorem 3.4, we have an exact sequence

$$0 \longrightarrow N \longrightarrow \text{Res}_{\beta_1, \dots, \beta_k} \text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1 \xrightarrow{\mathfrak{q}} \nabla_1 \boxtimes \dots \boxtimes \nabla_k \longrightarrow 0$$

for some submodule  $N$  of  $\text{Res}_{\beta_1, \dots, \beta_k} \text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1$ . Take  $\xi \in \text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1$  such that

$$\mathfrak{q}(\xi) = v_1 \otimes v_2 \otimes \dots \otimes v_k \in \nabla_1 \boxtimes \dots \boxtimes \nabla_k.$$

Let  $r_1 \otimes \dots \otimes r_k$  be an element of  $R(\beta_1) \otimes \dots \otimes R(\beta_k)$  such that  $\deg(r_1 \otimes \dots \otimes r_k) > 0$ . By (4.1), the element  $r_1 \otimes \dots \otimes r_k$  annihilates  $\nabla_1 \boxtimes \dots \boxtimes \nabla_k$ , which implies that

$$(4.3) \quad (r_1 \otimes \dots \otimes r_k) \xi \in N.$$

We now define a  $\mathbb{C}$ -linear map  $f \in (\text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1)^*$  by

$$f(\xi) = 1 \quad \text{and} \quad f(\zeta) = 0 \quad \text{for } \zeta \in N.$$

Note that  $f$  does not depend on the choice of  $\xi$ , and by (4.3)

$$(4.4) \quad \mathbb{C}f \cong \nabla_1 \boxtimes \dots \boxtimes \nabla_k.$$

On the other hand, by a direct computation, we may assume that

$$\xi = y \cdot v_k \otimes \dots \otimes v_1,$$

where  $y$  is the longest element in  $\Sigma_{\text{ht}(\beta)}/\Sigma_{\text{ht}(\beta_k)} \times \dots \times \Sigma_{\text{ht}(\beta_1)}$ . For any element

$$w \in \Sigma_{\text{ht}(\beta)}/\Sigma_{\text{ht}(\beta_k)} \times \dots \times \Sigma_{\text{ht}(\beta_1)},$$

there exists  $w' \in \Sigma_{\text{ht}(\beta)}$  such that  $w'w = y$ . Then, it follows from

$$(\psi(w')f)(x) = f(w'x) = \begin{cases} 1 & \text{if } x = w \cdot v_k \otimes \dots \otimes v_1, \\ 0 & \text{otherwise,} \end{cases}$$

that  $\{\psi(w')f \mid w \in \Sigma_{\text{ht}(\beta)}/\Sigma_{\text{ht}(\beta_k)} \times \dots \times \Sigma_{\text{ht}(\beta_1)}\}$  is a basis for  $(\text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1)^*$ . Hence the  $R(\beta)$ -module  $(\text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1)^*$  is generated by  $f$ .

Define the map

$$F : \nabla_1 \boxtimes \dots \boxtimes \nabla_k \longrightarrow \text{Res}_{\beta_1, \dots, \beta_k} (\text{Ind} \nabla_k \boxtimes \dots \boxtimes \nabla_1)^*$$

by mapping  $v_1 \otimes \cdots \otimes v_k$  to  $f$ . It follows from (4.4) that the map  $F$  is an  $R(\beta_1) \otimes \cdots \otimes R(\beta_k)$ -homomorphism. By Lemma 3.3, we have the  $R(\beta)$ -homomorphism

$$\mathcal{F} : \text{Ind} \nabla_1 \boxtimes \cdots \boxtimes \nabla_k \longrightarrow (\text{Ind} \nabla_k \boxtimes \cdots \boxtimes \nabla_1)^*$$

sending  $r \cdot v_1 \otimes \cdots \otimes v_k$  to  $r \cdot f$ . Since

$$\dim(\text{Ind} \nabla_1 \boxtimes \cdots \boxtimes \nabla_k) = \dim(\text{Ind} \nabla_k \boxtimes \cdots \boxtimes \nabla_1)^*$$

and  $(\text{Ind} \nabla_k \boxtimes \cdots \boxtimes \nabla_1)^*$  is generated by  $f$ , the map  $\mathcal{F}$  is an isomorphism, which proves the assertion (1).

The assertion (2) can be proved in a similar manner.  $\square$

**Lemma 4.3.**

- (1) For  $a \in \mathbb{Z}_{>0}$  and  $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k > 0$  with  $a + \ell_1 - 1 \leq n$ ,

$$\text{Ind} \nabla_{(a; \ell_1)} \boxtimes \nabla_{(a; \ell_2)} \boxtimes \cdots \boxtimes \nabla_{(a; \ell_k)}$$

is irreducible.

- (2) Let  $a_1, \dots, a_k \in \mathbb{Z}_{>0}$  and  $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_k > 0$ . Suppose that

$$a_i + \ell_i - 1 = a_j + \ell_j - 1 \leq n \quad (i \neq j).$$

Set  $b := a_1 + \ell_1 - 1$  and

$$M := \text{Ind} \nabla_{(a_1; \ell_1)} \boxtimes \nabla_{(a_2; \ell_2)} \boxtimes \cdots \boxtimes \nabla_{(a_k; \ell_k)}.$$

Then we have

- (a)  $M$  is irreducible,
- (b)  $\varepsilon_b(M) = k$ ,
- (c)  $\tilde{e}_b^k(M)$  is isomorphic to  $\text{Ind} \nabla_{(a_1; \ell_1 - 1)} \boxtimes \nabla_{(a_2; \ell_2 - 1)} \boxtimes \cdots \boxtimes \nabla_{(a_k; \ell_k - 1)}$ .

*Proof.* We first prove (2). We will use induction on  $\ell_1$ . If  $\ell_1 = 1$ , then our assertion follows from (3.3) immediately. Assume that  $\ell_1 > 1$ . Let

$$N := \text{Ind} \nabla_{(a_1; \ell_1 - 1)} \boxtimes \nabla_{(a_2; \ell_2 - 1)} \boxtimes \cdots \boxtimes \nabla_{(a_k; \ell_k - 1)}.$$

By the induction hypothesis,  $N$  is irreducible. By Lemma 3.3, it follows from

$$\text{Res}_{\alpha_{(a_i; \ell_i)} - \alpha_b, \alpha_b} \nabla_{(a_i; \ell_i)} \cong \nabla_{(a_i; \ell_i - 1)} \boxtimes \nabla_{(b; 1)}$$

that we get an exact sequence

$$\text{Ind} \nabla_{(a_i; \ell_i - 1)} \boxtimes \nabla_{(b; 1)} \longrightarrow \nabla_{(a_i; \ell_i)} \longrightarrow 0.$$

Since  $L(b^k) \cong \text{Ind} \underbrace{\nabla_{(b; 1)} \boxtimes \cdots \boxtimes \nabla_{(b; 1)}}_k$ , by transitivity of induction and Lemma 4.1 (2), we have

$$\text{Ind}(N \boxtimes L(b^k)) \cong \text{Ind}(N \boxtimes \underbrace{\nabla_{(b; 1)} \boxtimes \cdots \boxtimes \nabla_{(b; 1)}}_k) \longrightarrow M \longrightarrow 0.$$

Hence, from [16, Lemma 3.7], we conclude that

$$\varepsilon_b(\mathrm{hd}M) = k, \quad N \simeq \tilde{e}_b^k(\mathrm{hd}M),$$

and all the other composition factors  $L$  of  $M$  have  $\varepsilon_b(L) < k$ . On the other hand, from Lemma 4.2 and Lemma 4.1, we have

$$0 \longrightarrow \mathrm{hd}M \simeq (\mathrm{hd}M)^* \longrightarrow M^* \simeq M,$$

which yields  $\varepsilon_b(\mathrm{soc}M) \geq k$ . Therefore,  $M$  is irreducible.

Similarly, using the operator  $\tilde{e}_i^\vee$  in [19, (2.19)], one can prove the assertion (1).  $\square$

Combining Lemma 4.3 with (4.2) and Lemma 4.1, we obtain the following proposition.

**Proposition 4.4.** *Let  $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r > 0)$  be a Young diagram, and  $k \in \mathbb{Z}_{>0}$ . Assume that  $k + \mu_1 - 1 \leq n$ . Then*

- (1)  $\mathrm{Ind}\nabla_\mu[k]$  is irreducible,
- (2)  $\mathrm{Ind}\nabla_\mu[k]$  is isomorphic to  $\mathrm{Ind}^t\nabla_\mu[k]$ .

Let  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s > 0)$  be a Young diagram and let  $\Psi_\lambda : \mathbf{B}(\lambda) \longrightarrow \mathcal{Y}^s$  be the injective map defined by (1.2). For a semistandard tableau  $T$  of shape  $\lambda$ , define

$$(4.5) \quad \nabla_T := \nabla_{\mu^{(s)}}[s] \boxtimes \nabla_{\mu^{(s-1)}}[s-1] \boxtimes \dots \boxtimes \nabla_{\mu^{(1)}}[1],$$

where  $\Psi_\lambda(T) = (\mu^{(1)}, \dots, \mu^{(s)})$ .

Let  $\mu = (\mu_1 \geq \dots \geq \mu_r > 0)$  be a Young diagram, and

$${}^t\mu = (c_1 \geq \dots \geq c_t > 0).$$

For  $k \in \mathbb{Z}_{>0}$ , define

$$\mathbf{i}(\mu; k) := (\underbrace{k, \dots, k}_{c_1}, \underbrace{k+1, \dots, k+1}_{c_2}, \dots, \underbrace{k+\mu_1-1, \dots, k+\mu_1-1}_{c_t}) \in I^{\alpha_\mu[k]}.$$

If  $k + \mu_1 - 1 \leq n$ , then it follows from Proposition 3.1 and Proposition 4.4 that

$$(4.6) \quad \mathbf{i}(\mu; k) \text{ occurs in } \mathrm{ch}(\mathrm{Ind}\nabla_\mu[k]) \text{ with multiplicity } {}^t\mu! := c_1!c_2!\dots c_t!.$$

By Proposition 3.1 and (4.6), we deduce

$$(4.7) \quad \mathbf{i}(\mu^{(s)}; s) * \dots * \mathbf{i}(\mu^{(1)}; 1) \text{ occurs in } \mathrm{ch}(\mathrm{Ind}\nabla_T) \text{ with multiplicity } {}^t\mu^{(s)}! \dots {}^t\mu^{(1)}!.$$

Now we will state and prove one of our main results.

**Theorem 4.5.** *Let  $T$  be a semistandard tableau of shape  $\lambda$ . Then  $\mathrm{hd}\mathrm{Ind}\nabla_T$  is irreducible.*

*Proof.* Let  $\Psi_\lambda(T) = (\mu^{(1)}, \dots, \mu^{(s)})$  and let  $Q$  be a quotient of  $\mathrm{Ind}T$ . It follows from Proposition 4.4 that  $(\mathrm{Ind}\nabla_{\mu^{(s)}}[s]) \boxtimes (\mathrm{Ind}\nabla_{\mu^{(s-1)}}[s-1]) \boxtimes \dots \boxtimes (\mathrm{Ind}\nabla_{\mu^{(1)}}[1])$  is irreducible. Then, by Lemma 3.3, we have the following exact sequence

$$0 \longrightarrow (\mathrm{Ind}\nabla_{\mu^{(s)}}[s]) \boxtimes (\mathrm{Ind}\nabla_{\mu^{(s-1)}}[s-1]) \boxtimes \dots \boxtimes (\mathrm{Ind}\nabla_{\mu^{(1)}}[1]) \longrightarrow \mathrm{Res}_{\alpha_{\mu^{(s)}}[s], \dots, \alpha_{\mu^{(1)}}[1]} Q,$$

which implies that, by (4.6) and (4.7),

$$\mathbf{i}(\mu^{(s)}; s) * \cdots * \mathbf{i}(\mu^{(1)}; 1) \text{ occurs in } \text{ch}Q \text{ with multiplicity } {}^t\mu^{(s)}! \cdots {}^t\mu^{(1)}!.$$

Therefore, our assertion follows from Lemma 3.11.  $\square$

Thus we obtain a map  $\mathbf{B}(\lambda) \rightarrow \mathfrak{B}(\infty) \otimes \mathbf{T}^\lambda \otimes \mathbf{C}$  given by

$$T \mapsto \text{hdInd} \nabla_T \otimes t_\lambda \otimes c \quad (T \in \mathbf{B}(\lambda)).$$

We will show that this map is the strict crystal embedding which maps the maximal vector  $T_\lambda$  to  $\mathbf{1} \otimes t_\lambda \otimes c$ . Here,  $\mathbf{1}$  is the trivial  $R(0)$ -module.

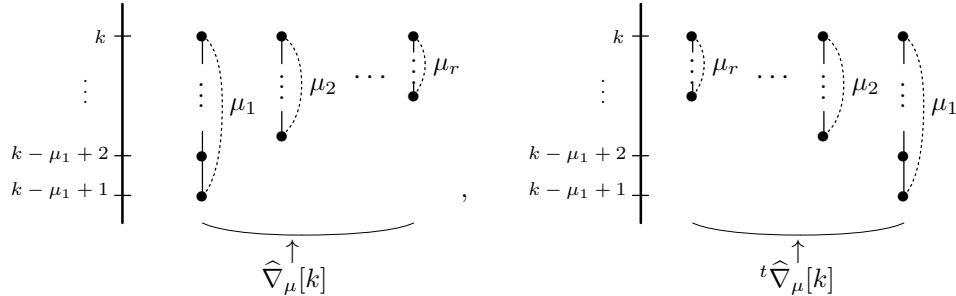
For a Young diagram  $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r > 0)$ , let

$$\mu^+ := (\mu_1 - 1 \geq \mu_2 - 1 \geq \cdots \geq \mu_r - 1 \geq 0).$$

For  $k = 1, \dots, n$  with  $k - \mu_1 + 1 \geq 1$ , we define

$$\begin{aligned} \widehat{\nabla}_\mu[k] &:= \nabla_{(k-\mu_1+1; \mu_1)} \boxtimes \nabla_{(k-\mu_2+1; \mu_2)} \boxtimes \cdots \boxtimes \nabla_{(k-\mu_r+1; \mu_r)}, \\ {}^t\widehat{\nabla}_\mu[k] &:= \nabla_{(k-\mu_r+1; \mu_r)} \boxtimes \nabla_{(k-\mu_{r-1}+1; \mu_{r-1})} \boxtimes \cdots \boxtimes \nabla_{(k-\mu_1+1; \mu_1)}. \end{aligned}$$

Pictorially, the modules  $\widehat{\nabla}_\mu[k]$  and  ${}^t\widehat{\nabla}_\mu[k]$  may be visualized as follows:



By Lemma 4.3 and Lemma 4.1, we have the following lemma.

**Lemma 4.6.** *Let  $\mu = (\mu_1 \geq \cdots \geq \mu_r > 0)$  and  $k = 1, \dots, n$  with  $k - \mu_1 + 1 \geq 1$ .*

- (1)  $\text{Ind} \widehat{\nabla}_\mu[k]$  is irreducible.
- (2)  $\text{Ind} \widehat{\nabla}_\mu[k]$  is isomorphic to  $\text{Ind} {}^t\widehat{\nabla}_\mu[k]$ .
- (3)  $\varepsilon_k(\text{Ind} \widehat{\nabla}_\mu[k]) = r$ .
- (4)  $\tilde{e}_k^r(\text{Ind} \widehat{\nabla}_\mu[k]) \simeq \text{Ind} \widehat{\nabla}_{\mu^+}[k-1]$ .

Let  $T$  be a semistandard tableau of shape  $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s > 0)$ . Suppose that  $T$  is not the maximal vector  $T_\lambda$ . Write  $\Psi_\lambda(T) = (\mu^{(1)}, \dots, \mu^{(s)})$  and

$$\mu^{(i)} = (\mu_1^{(i)} \geq \mu_2^{(i)} \geq \cdots \geq \mu_{\lambda_i}^{(i)} \geq 0) \quad (i = 1, \dots, s).$$

Recall the notations given in Lemma 1.4:

$$i_T := \min\{\mu_j^{(i)} + i - 1 \mid 1 \leq i \leq s, 1 \leq j \leq \lambda_i, \mu_j^{(i)} > 0\},$$

$$T^+ := \text{the tableau of shape } \lambda \text{ obtained from } T \text{ by replacing the entries } i_T + 1 \text{ by } i_T \text{ from the top row to the } i_T\text{th row.}$$

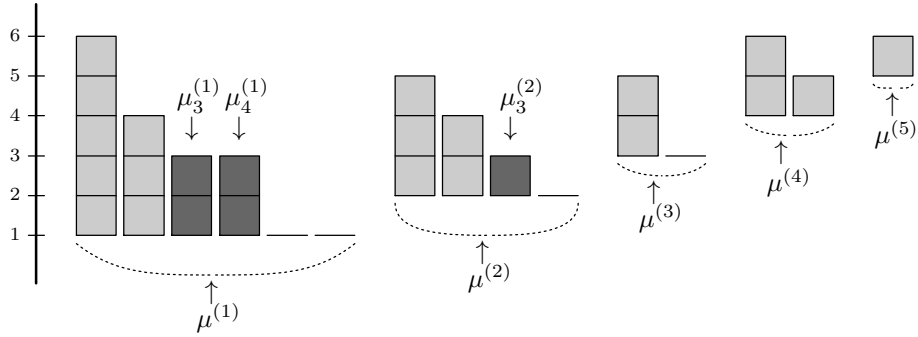
We define

$$\begin{aligned} \overline{\mu}^{(i)} &:= (\mu_j^{(i)} \mid \mu_j^{(i)} + i - 1 \neq i_T) \quad \text{for } 1 \leq i \leq s, \\ \mu_{\min} &:= (\mu_j^{(i)} \mid \mu_j^{(i)} + i - 1 = i_T, 1 \leq i \leq s). \end{aligned}$$

Note that  $\mu_{\min}$  is not the empty Young diagram  $(0, 0, \dots)$  and, by construction, for any component  $\mu_j^{(i)}$  in  $\mu_{\min}$ , we have

$$(4.8) \quad \mu_{j'}^{(i)} \geq \mu_j^{(i)} \quad \text{for } j' \leq j, \quad \mu_{j'}^{(i')} + i' \geq \mu_j^{(i)} + i \quad \text{for } i' < i.$$

**Example 4.7.** We use the same notations as in Example 1.3 and Example 1.5. Consider the following diagram for  $\Psi_\lambda(T)$ :

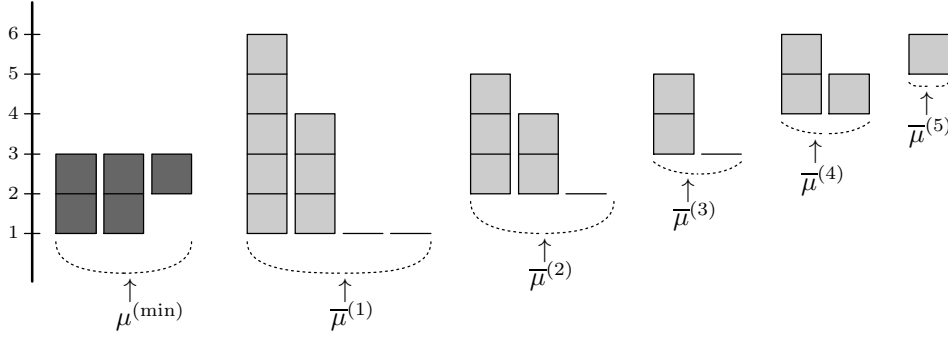


Then we have

$$\begin{aligned} \mu_{\min} &= (\mu_3^{(1)}, \mu_4^{(1)}, \mu_3^{(2)}), \\ \overline{\mu}^{(1)} &= (\mu_1^{(1)}, \mu_2^{(1)}, \mu_5^{(1)}, \mu_6^{(1)}), \quad \overline{\mu}^{(2)} = (\mu_1^{(2)}, \mu_2^{(2)}, \mu_4^{(2)}), \\ \overline{\mu}^{(3)} &= (\mu_1^{(3)}, \mu_2^{(3)}), \quad \overline{\mu}^{(4)} = (\mu_1^{(4)}, \mu_2^{(4)}), \quad \overline{\mu}^{(5)} = (\mu_1^{(5)}). \end{aligned}$$

Pictorially, the partitions  $\mu_{\min}$  and  $\overline{\mu}^{(i)}$  ( $i = 1, \dots, 5$ ) are given as follows:





By Proposition 4.4, Lemma 4.1 (2) and (4.8), we obtain

$$\begin{aligned}
 \text{Ind} \nabla_T &\simeq \text{Ind}(\nabla_{\mu^{(s)}}[s] \boxtimes \cdots \boxtimes \nabla_{\mu^{(1)}}[1]) \\
 &\simeq \text{Ind}({}^t \nabla_{\mu^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\mu^{(1)}}[1]) \\
 (4.9) \quad &\simeq \text{Ind}({}^t \nabla_{\bar{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\bar{\mu}^{(1)}}[1] \boxtimes {}^t \widehat{\nabla}_{\mu_{\min}}[i_T]).
 \end{aligned}$$

In the same manner, we have

$$(4.10) \quad \text{Ind} \nabla_{T^+} \simeq \text{Ind}({}^t \nabla_{\bar{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\bar{\mu}^{(1)}}[1] \boxtimes {}^t \widehat{\nabla}_{\mu_{\min}^+}[i_T - 1]).$$

Now, we will prove our main result.

**Theorem 4.8.**

- (1) For  $T \in \mathbf{B}(\lambda)$ ,  $\text{hdInd} \nabla_T$  is an irreducible  $R^\lambda$ -module.
- (2) The map  $\Phi_\lambda : \mathbf{B}(\lambda) \rightarrow \mathfrak{B}(\lambda)$  defined by

$$\Phi_\lambda(T) = \text{hdInd} \nabla_T \quad (T \in \mathbf{B}(\lambda))$$

is a crystal isomorphism.

*Proof.* Let  $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s > 0)$ , and let

$$\begin{aligned}
 \phi_\lambda : \mathbf{B}(\lambda) &\rightarrow \mathfrak{B}(\infty) \otimes \mathbf{T}^\lambda \otimes \mathbf{C}, \\
 T &\mapsto \text{hdInd} \nabla_T \otimes t_\lambda \otimes c.
 \end{aligned}$$

We first show that  $\phi_\lambda$  is the strict crystal embedding which maps the maximal vector  $T_\lambda$  to  $\mathbf{1} \otimes t_\lambda \otimes c$ . Here,  $\mathbf{1}$  is the trivial  $R(0)$ -module. It is obvious that  $\phi_\lambda$  maps  $T_\lambda$  to  $\mathbf{1} \otimes t_\lambda \otimes c$ . If  $T_\lambda = \tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_k}^{\max} T$  for  $T \in \mathbf{B}(\lambda)$  and  $i_j \in I$ , then it suffices to show that

$$\tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_k}^{\max} \phi_\lambda(T) = \phi_\lambda(\tilde{e}_{i_1}^{\max} \cdots \tilde{e}_{i_k}^{\max} T).$$

We will use induction on  $\text{ht}(\lambda - \text{wt}(T))$ . If  $\text{wt}(T) = \lambda$ , then there is nothing to prove. Assume that  $\text{ht}(\lambda - \text{wt}(T)) > 0$ . Write  $\Psi_\lambda(T) = (\mu^{(1)}, \dots, \mu^{(s)})$  and

$$\mu^{(i)} = (\mu_1^{(i)} \geq \mu_2^{(i)} \geq \cdots \geq \mu_{\lambda_i}^{(i)} \geq 0) \quad (i = 1, \dots, s).$$

From (4.9), we obtain

$$\mathrm{Ind} \nabla_T \simeq \mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1] \boxtimes {}^t \widehat{\nabla}_{\mu_{\min}}[i_T]).$$

Let

$$\varepsilon := \varepsilon_{i_T}(\mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}}[i_T])) \in \mathbb{Z}_{>0}.$$

Since  $\Delta_{i_T}(\nabla_{(i; \mu_j^{(i)})}) = 0$  for any  $\mu_j^{(i)} \in \overline{\mu}^{(i)}$ , it follows from Proposition 3.1 that

$$\varepsilon_{i_T}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1])) = 0.$$

By Lemma 3.3, we have the following nontrivial map:

$$(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s]) \boxtimes \cdots \boxtimes (\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(1)}}[1]) \boxtimes (\mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}}[i_T])) \longrightarrow \mathrm{Res}(\mathrm{hd} \mathrm{Ind} \nabla_T),$$

which implies that, by Proposition 4.4 and Lemma 4.6,

$$\varepsilon = \varepsilon_{i_T}(\mathrm{hd} \mathrm{Ind} \nabla_T).$$

Hence, by Lemma 3.9, we have

$$[e_{i_T}^\varepsilon(\mathrm{hd} \mathrm{Ind} \nabla_T)] = q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\tilde{e}_{i_T}^\varepsilon(\mathrm{hd} \mathrm{Ind} \nabla_T)].$$

On the other hand, by Lemma 3.5, we obtain

$$\begin{aligned} e_{i_T}(\mathrm{Ind} \nabla_T) &\simeq e_{i_T}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1] \boxtimes {}^t \widehat{\nabla}_{\mu_{\min}}[i_T])) \\ &\simeq \mathrm{Ind}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1]) \boxtimes e_{i_T} \mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}}[i_T])). \end{aligned}$$

It follows from Lemma 3.9, Lemma 4.6 and (4.10) that

$$\begin{aligned} [e_{i_T}^\varepsilon(\mathrm{Ind} \nabla_T)] &\simeq [\mathrm{Ind}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1]) \boxtimes e_{i_T}^\varepsilon \mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}}[i_T]))] \\ &\simeq q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\mathrm{Ind}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1]) \boxtimes \tilde{e}_{i_T}^\varepsilon \mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}}[i_T]))] \\ &\simeq q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\mathrm{Ind}(\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1]) \boxtimes \mathrm{Ind}({}^t \widehat{\nabla}_{\mu_{\min}^+}[i_T - 1]))] \\ &\simeq q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\mathrm{Ind}({}^t \nabla_{\overline{\mu}^{(s)}}[s] \boxtimes \cdots \boxtimes {}^t \nabla_{\overline{\mu}^{(1)}}[1] \boxtimes {}^t \widehat{\nabla}_{\mu_{\min}^+}[i_T - 1])] \\ &\simeq q^{-\frac{\varepsilon(\varepsilon-1)}{2}} [\varepsilon]_q! [\mathrm{Ind} \nabla_{T+}]. \end{aligned}$$

Since  $e_{i_T}$  is an exact functor, we obtain an exact sequence

$$e_{i_T}^\varepsilon(\mathrm{Ind} \nabla_T) \longrightarrow e_{i_T}^\varepsilon(\mathrm{hd} \mathrm{Ind} \nabla_T) \longrightarrow 0,$$

which yields that  $\mathrm{hd} \mathrm{Ind} \nabla_{T+} \simeq \tilde{e}_{i_T}^\varepsilon(\mathrm{hd} \mathrm{Ind} \nabla_T)$ . By Lemma 1.4, we conclude

$$\begin{aligned} \phi_\lambda(\tilde{e}_{i_T}^\varepsilon T) &= \mathrm{hd} \mathrm{Ind}(\nabla_{\tilde{e}_{i_T}^\varepsilon T}) \otimes t_\lambda \otimes c \\ &\simeq \mathrm{hd} \mathrm{Ind} \nabla_{T+} \otimes t_\lambda \otimes c \\ &\simeq \tilde{e}_{i_T}^\varepsilon(\mathrm{hd} \mathrm{Ind} \nabla_T) \otimes t_\lambda \otimes c \\ &= \tilde{e}_{i_T}^\varepsilon \phi_\lambda(T). \end{aligned}$$

By induction hypothesis,  $\phi_\lambda$  is the strict crystal embedding. Therefore, our assertions (1) and (2) follow from the crystal embedding  $\mathfrak{B}(\lambda) \rightarrow \mathfrak{B}(\infty) \otimes \mathbf{T}^\lambda$  ( $M \mapsto \mathrm{infl}_\lambda M \otimes t_\lambda$ ) given in [19, (5.10)].

□

As a result, the set  $\{\text{hdInd}\nabla_T \mid T \in \mathbf{B}(\lambda)\}$  gives a complete list of irreducible graded  $R$ -modules up to isomorphism and grading shift.

Let us construct the inverse morphism  $\Theta_\lambda : \mathfrak{B}(\lambda) \rightarrow \mathbf{B}(\lambda)$  of  $\Phi_\lambda$ . The following lemma is crucial.

**Lemma 4.9.** *Let  $M$  be an irreducible graded  $R^\lambda(\alpha)$ -module and let  $T$  be a semistandard tableau in  $\mathbf{B}(\lambda)$ . Then the following are equivalent.*

- (1)  $M$  is isomorphic to  $\text{hdInd}\nabla_T$ .
- (2)  $\text{wt}(T) = \lambda - \alpha$  and  $\dim \text{HOM}(\nabla_T, \text{Res}M) \neq 0$ .

*Proof.* Assume that  $M$  is isomorphic to  $\text{hdInd}\nabla_T$ . Clearly,  $\text{wt}(T) = \lambda - \alpha$ . Moreover, by Lemma 3.3, we have

$$\dim \text{HOM}(\nabla_T, \text{Res}M) \neq 0.$$

Conversely, suppose that  $\text{wt}(T) = \lambda - \alpha$  and  $\dim \text{HOM}(\nabla_T, \text{Res}M) \neq 0$ . Then we have a nontrivial map

$$\text{Ind}\nabla_T \longrightarrow M \longrightarrow 0.$$

Then it follows from Theorem 4.5 that  $M \cong \text{hdInd}\nabla_T$ . □

Given an irreducible  $R^\lambda(\alpha)$ -module  $M$ , we take  $T_M \in \mathbf{B}(\lambda)$  such that

$$\text{wt}(T_M) = \lambda - \alpha \quad \text{and} \quad \dim \text{HOM}(\nabla_{T_M}, \text{Res}M) \neq 0.$$

By Theorem 4.8 and Lemma 4.9, the tableau  $T_M$  is well-defined. Now, it is straightforward to verify that  $\Phi_\lambda$  and  $\Theta_\lambda$  are inverses to each other.

**Proposition 4.10.** *The map defined by  $\Theta_\lambda : \mathfrak{B}(\lambda) \rightarrow \mathbf{B}(\lambda)$  defined by*

$$\Theta_\lambda(M) = T_M \quad (M \in \mathfrak{B}(\lambda))$$

*is the inverse morphism of  $\Phi_\lambda$ .*

## 5. IRREDUCIBLE $R$ -MODULES AND marginally large TABLEAUX

In this section, using the results proved in Section 4, we construct an explicit crystal isomorphism  $\Phi_\infty : \mathbf{B}(\infty) \xrightarrow{\sim} \mathfrak{B}(\infty)$ . Consequently, we obtain a complete list of irreducible graded  $R$ -modules up to isomorphism and grading shift.

Let us recall the map  $\Psi_\infty : \mathbf{B}(\infty) \rightarrow \mathcal{Y}^n$ ,  $T \mapsto (\mu^{(1)}, \dots, \mu^{(n)})$  defined in (2.2). For  $T \in \mathbf{B}(\infty)$ , we define

$$\nabla_T := \nabla_{\mu^{(n)}}[n] \boxtimes \nabla_{\mu^{(n-1)}}[n-1] \boxtimes \cdots \boxtimes \nabla_{\mu^{(1)}}[1].$$

By Lemma 2.4, if  $\iota_\lambda(T') = T \otimes t_\lambda \otimes c$  for  $T' \in \mathbf{B}(\lambda)$ , then we have

$$(5.1) \quad \nabla_{T'} = \nabla_T.$$

**Theorem 5.1.** *The map  $\Phi_\infty : \mathbf{B}(\infty) \rightarrow \mathfrak{B}(\infty)$  defined by*

$$\Phi_\infty(T) := \text{hdInd} \nabla_T \quad (T \in \mathbf{B}(\infty))$$

*is a crystal isomorphism.*

*Proof.* It is obvious that  $\Phi_\infty$  maps the maximal vector  $T_\infty$  of  $\mathbf{B}(\infty)$  to the maximal vector  $\mathbf{1}$  of  $\mathfrak{B}(\infty)$ . Here,  $\mathbf{1}$  is the trivial  $R(0)$ -module. Take a tableau  $T \in \mathbf{B}(\infty)$  and suppose that  $T_\infty = \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} T$  for some  $i_k = 1, \dots, n$ ,  $j_k \in \mathbb{Z}_{>0}$ . Then it suffices to show that

$$\tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} \Phi_\infty(T) = \Phi_\infty(\tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} T).$$

Take a dominant integral weight  $\lambda \in P^+$  with  $\lambda(h_i) \gg 0$  for all  $i \in I$  so that one can find  $T' \in \mathbf{B}(\lambda)$  satisfying

$$\iota_\lambda(T') = T \otimes t_\lambda \otimes c,$$

where  $\iota_\lambda : \mathbf{B}(\lambda) \rightarrow \mathbf{B}(\infty) \otimes \mathbf{T}^\lambda \otimes \mathbf{C}$  is the crystal embedding given in (2.1). Note that  $T_\lambda = \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} T'$ . Hence it follows from Theorem 4.8 and (5.1) that

$$\begin{aligned} \Phi_\infty(\tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} T) &= \Phi_\infty(T_\infty) = \Phi_\lambda(T_\lambda) = \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} \Phi_\lambda(T') \\ &= \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} (\text{hdInd} \nabla_{T'}) = \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} (\text{hdInd} \nabla_T) \\ &= \tilde{e}_{i_1}^{j_1} \tilde{e}_{i_2}^{j_2} \cdots \tilde{e}_{i_t}^{j_t} \Phi_\infty(T), \end{aligned}$$

which completes the proof.  $\square$

We now construct the inverse map

$$\Theta_\infty : \mathfrak{B}(\infty) \rightarrow \mathbf{B}(\infty)$$

of the crystal isomorphism  $\Phi_\infty$ . Given an irreducible  $R(\alpha)$ -module  $M$ , we take  $T_M \in \mathbf{B}(\infty)$  such that

$$\text{wt}(T_M) = -\alpha \quad \text{and} \quad \dim \text{HOM}(\nabla_{T_M}, \text{Res} M) \neq 0.$$

By Lemma 4.9 and Theorem 5.1, we obtain the following proposition.

**Proposition 5.2.** *The map  $\Theta_\infty : \mathfrak{B}(\infty) \rightarrow \mathbf{B}(\infty)$  defined by*

$$\Theta_\infty(T) = T_M \quad (T \in \mathfrak{B}(\infty))$$

*is the inverse morphism of  $\Phi_\infty$ .*

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